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On the Solvability of an Initial-Boundary Value Problem for a Fractional Heat Equation with Involution

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Abstract—The present work is devoted to the study of methods for solving the Dirichlet boundary value problem for a class of nonlocal second-order partial differential equations with involutive mappings of arguments. The concept of a nonlocal analogue of the Laplace equation is introduced, which generalizes the classical Laplace equation. The problems are solved by applying the theory of matrices and the method of separation of variables. Research of the substantiation of the well-posedness of these problems is carried out, as well as the proof of existence and uniqueness theorems for solutions of the corresponding boundary value problems. Authors proposed the method that allows, using the theory of matrices, to reduce the study of a boundary value problem to another problem for a parabolic equation without involution.

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1. INTRODUCTION AND PROBLEM STATEMENT

Although nonlocal equations have been studied for a long time (see [1]), the concept of a nonlocal operator and the associated concept of a nonlocal differential equation arose in mathematics relatively recently. At present, nonlocal differential equations include equations in which the unknown function and its derivatives enter, generally speaking, for different values of the arguments. Examples of such equations are loaded equations, fractional integro-differential equations, equations with deviating arguments, etc. [2]. Such nonlocal equations arise in some problems of integral geometry, inverse problems of kinematic seismics and geophysics, vibration problems, problems of the elasticity theory, the theory of magnetohydrodynamic flows, the theory of propagation of elastic electromagnetic waves described by Maxwell's equation with memory, the plasticity and creep theory, and others (see, e.g., [3–6]).

Among differential equations with deviating arguments, a special place is occupied by equations with alternating signs of the arguments deviation. One such deviation is the so-called deviation of involutive type that arises in filtration theory, forecasting theory, as well as in the study of subharmonic oscillations (see [7–10]). There has been extensive research in this area, so we note only a few papers that are close to our study.

Differential operators of the first and second order with involution, as well as their spectral decompositions, were studied in [11–14], and the properties of the basis property of root functions were considered in [15–18].

Spectral properties of operators (asymptotics of eigenvalues, equiconvergence estimates for spectral expansions) for a system of differential equations with involution were studied by the method of similar operators in [19, 20].

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In this paper, we study the Dirichlet problem for a nonlocal analogue of a parabolic fractional-order equation with three independent variables. Note that various boundary value problems for parabolic equations were studied in [21–24].

Let p, q and T be positive real numbers, $\Pi = \{x = (x_1, x_2) : 0 < x_1 < p; 0 < x_1 < q\}$, $\Omega = (0, T) \times \Pi$, and let $\partial\Pi$ be the boundary of the region Π. We consider the mappings

$$S_0 x = (x_1, x_2), \quad S_1 x = (p - x_1, x_2), \quad S_2 x = (x_1, q - x_2), \quad S_3 x = (p - x_1, q - x_2),$$

defined for all points $x = (x_1, x_2) \in \Pi$. Obviously, the equalities $S_j^2 x = x$ hold for $j = \overline{0, 3}$, i.e. the mappings S_j are involutions. In addition, the following equalities hold true:

$$S_1 \cdot S_2 = S_2 \cdot S_1 = S_3, \quad S_1 \cdot S_3 = S_3 \cdot S_1 = S_2, \quad S_2 \cdot S_3 = S_3 \cdot S_2 = S_1.$$

Let $a_j,\ j = \overline{0, 3}$, be real numbers and Δ be the Laplace operator with respect to variables x_1 and x_2. For functions $v(x_1, x_2) \in C^2(\Pi)$, we introduce the operator

$$Lv(x) \equiv a_0 \Delta v(S_0 x) + a_1 \Delta v(S_1 x) + a_1 \Delta v(S_2 x) + a_1 \Delta v(S_3 x).$$

We will refer to L as the nonlocal Laplace operator. In the case $a_0 = 1$, $a_j = 0$, $j = 1, 2, 3$, the operator L coincides with the standard two-dimensional Laplace operator.

The integration operator of order $\alpha > 0$ in the sense of Hadamard [25] is defined as

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} y(\tau) \frac{d\tau}{\tau}, \quad t > 0.$$

If $\alpha = 0$ we assume that $J^0 y(t) = y(t)$. For $\alpha \in (m-1, m], m = 1, 2, ...$, the expression

$$D^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \left(\ln \frac{t}{\tau}\right)^{\alpha-m-1} \delta^m y(\tau) \frac{d\tau}{\tau},$$

where $\delta = t\frac{d}{dt}$ and $\delta^k = \delta \cdot \delta^{k-1}, k \geq 1$, is called the differentiation operator of order $\alpha > 0$ in the sense of Hadamard–Caputo [25].

Let $0 < \alpha \leq 1$, $\beta > 0$ and $\Delta u(t, S_j x)$ stand for $\Delta u(t, S_j x) = \Delta u(t, z)|_{z = S_j x}$, $j = \overline{0, 3}$. Let us consider the following equation in the domain Ω:

$$t^{-\beta} D_t^\alpha u(t, x) = Lu(t, x), (t, x) \in \Omega. \tag{1}$$

In the case $\alpha = 1, \beta = 1$, we have $t^{-\beta} D_t^\alpha u(t, x) = t^{-1}\left(t\frac{\partial}{\partial t}\right) u(t, x) = \frac{\partial u(t,x)}{\partial t}$. Hence, for $a_0 = 1, a_j = 0, j = 1, 2, 3$, equation (1) coincides with the classical parabolic equation, whereas for $\alpha = 1, a_j \neq 0, j = \overline{0, 3}$, we get a nonlocal analogue of the parabolic equation. Thus, equation (1) is a fractional analogue of a nonlocal parabolic equation with three independent variables.

We consider the following problem for equation (1) in the domain Ω.

Problem D. Find a function $u(t, x)$ from the class $C(\bar{\Omega}) \cap C^{\alpha,2,2}_{t,x_1,x_2}(\Omega)$ that satisfies equation (1) and initial and boundary conditions

$$u(0, x) = \varphi(x), \quad x \in \bar{\Pi}, \tag{2}$$

$$u(t, x) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Pi, \tag{3}$$

where $\varphi(x)$ is a given function; $u \in C_t^\alpha(\Omega) \overset{def}{\Leftrightarrow} D_t^\alpha u \in C(\Omega)$.

2. THE CAUCHY PROBLEM FOR A ONE-DIMENSIONAL FRACTIONAL DIFFERENTIAL EQUATION

In this section, we study the Cauchy problem for a one-dimensional fractional differential equation with the Hadamard derivative.

Let $0 < \alpha \leq 1$ and $\beta > 0$. We introduce the operators

$$B_\alpha^\beta y(t) = t^{-\beta} D_t^\alpha y(t), \quad B_\alpha^{-\beta} y(t) = J^\alpha[\tau^\beta y](t)$$

and present some their properties.

It can be easily shown that the following identities hold true:

$$J^\alpha(t^\mu) = \mu^{-\alpha} t^\mu, \quad \text{for} \quad \alpha > 0, \quad \mu > 0,$$

$$D^\alpha(t^\mu) = \left\{ \begin{cases} 0, & \mu = 0, \\ \mu^\alpha t^\mu, & \mu > 0, \end{cases} \right. \quad \text{for} \quad \alpha \in (0, 1]. \tag{4}$$

Lemma 1. *If $f(t) \in C[0, d]$, then $B_\alpha^{-\beta} y(t) \in [0, d]$ and the following estimate holds*:

$$\left|\left|B_\alpha^{-\beta} f(t)\right|\right|_{C[0,d]} \leq \frac{d^\beta}{\beta^\alpha} ||f(\tau)||_{C[0,d]}.$$

Proof. Let $f(t) \in C[0, d]$. Then, by the definition of the operator $B_\alpha^{-\beta}$, we have

$$\left|B_\alpha^{-\beta} f(t)\right| = \left|\frac{1}{\Gamma(\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\beta}}\right| \leq \frac{||f(\tau)||_{C[0,d]}}{\Gamma(\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \tau^{\beta-1} d\tau$$

$$= \left(\xi = \ln \frac{t}{\tau}, \tau = te^{-\xi}\right) = \frac{||f(\tau)||_{C[0,d]}}{\Gamma(\alpha)} t^\beta \int\limits_0^\infty \xi^{\alpha-1} e^{-\beta\xi} d\xi = (\tau = \beta\xi)$$

$$= \frac{||f(\tau)||_{C[0,d]}}{\Gamma(\alpha)} \frac{t^\beta}{\beta^\alpha} \int\limits_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau = \frac{||f(\tau)||_{C[0,d]}}{\beta^\alpha} t^\beta.$$

Hence, the following estimate holds:

$$\left|B_\alpha^{-\beta} f(t)\right| \leq \frac{||f(\tau)||_{C[0,d]}}{\beta^\alpha} t^\beta.$$

Consequently, we have

$$\left|\left|B_\alpha^{-\beta} f(t)\right|\right|_{C[0,d]} = \max_{0 \leq t \leq d} \left|B_\alpha^{-\beta} f(t)\right| \leq \frac{||f(\tau)||_{C[0,d]}}{\beta^\alpha} \max_{0 \leq t \leq d} t^\beta = \frac{d^\beta}{\beta^\alpha} ||f(\tau)||_{C[0,d]}.$$

The lemma 1 is proved. □

Lemma 2. *Let $0 < \alpha \leq 1$ and $f(t) \in C^1[0, d]$. Then $D^\alpha y(t) \in [0, d]$, $D^\alpha y(0) = 0$ and the following estimate holds*:

$$||D^\alpha f(t)||_{C[0,d]} \leq d||f'(\tau)||_{C[0,d]}.$$

Proof. If $\alpha = 1$, then $D^1 f(t) = t\frac{d}{dt} f(t)$. Hence, for $f(t) \in C^1[0, d]$ we have $D^1 f(t) \in [0, d]$. Obviously, $D^1 f(t)\big|_{t=0} = t\frac{d}{dt} f(t)\big|_{t=0} = 0$. Let $0 < \alpha < 1$. Then

$$|D^\alpha f(t)| = \left|\frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} \delta f(\tau) \frac{d\tau}{\tau}\right| \leq \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} |f'(\tau)| d\tau$$

$$\leq ||f'(\tau)||_{C[0,d]} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau}\right)^{-\alpha} d\tau = \left(\xi = \ln \frac{t}{\tau}, \tau = te^{-\xi}\right)$$

$$= ||f'(\tau)||_{C[0,d]} \frac{t}{\Gamma(1-\alpha)} \int_0^t \xi^{-\alpha} e^{-\xi} d\tau = t||f'(\tau)||_{C[0,d]}.$$

Hence, we obtain

$$||D^\alpha f(t)||_{C[0,d]} \leq d||f'(\tau)||_{C[0,d]} \quad \text{and} \quad D^\alpha f(t)|_{t=0} = \lim_{t\to 0} D^\alpha f(t) = 0.$$

The lemma 2 is proved. □

Lemma 3. *Let* $0 < \alpha \leq 1$ *and* $f(t) \in C^1[0, d]$. *Then the following identity is true*:

$$B_\alpha^{-\beta}\left[B_\alpha^\beta[f]\right](t) = f(t) - f(0). \tag{5}$$

Proof. If $f(t) \in C^1[0, d]$, then it follows from Lemma 2 that the function $B_\alpha^\beta[f](t)$ belongs to the class $C(0, d]$. If $\alpha = 1$, then we have

$$B_1^{-\beta}\left[B_1^\beta[f]\right](t) = \frac{1}{\Gamma(\alpha)} \int_0^t \tau^\beta B_1^\beta[f](\tau)\frac{d\tau}{\tau} = \frac{1}{\Gamma(\alpha)} \int_0^t \tau \frac{d}{d\tau} f(\tau)\frac{d\tau}{\tau} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d}{d\tau} f(\tau)d\tau = f(t) - f(0).$$

For $0 < \alpha < 1$ we obtain

$$B_\alpha^{-\beta} B_\alpha^\beta[f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} \tau^\beta B_\alpha^\beta[f](\tau)\frac{d\tau}{\tau} = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} D^\alpha[f](\tau)\frac{d\tau}{\tau}$$
$$= J^\alpha\left[J^{1-\alpha}[\tau f']\right](t) = J^1[f'](t) = f(t) - f(0).$$

The lemma 3 is proved. □

In the domain $(0, d)$, we consider the following Cauchy problem:

$$t^{-\beta} D^\alpha y(t) = \lambda y(t), \quad 0 < t < d, \tag{6}$$

$$y(0) = b, \tag{7}$$

where b is a positive real number. We will call as a solution to problem (6), (7) the function $y(t) \in C[0, d]$ such that $D^\alpha y(t) \in C(0, d)$ and satisfies (6) and (7) in the classical sense.

Let a function $y(t)$ be a solution to problem (6), (7). By applying the operator $B_\alpha^{-\beta}$ to both sides of equation (6), we have

$$B_\alpha^{-\beta}\left[t^{-\beta} D^\alpha y\right](t) = \lambda B_\alpha^{-\beta}[y](t), \quad 0 < t < d.$$

From (5), taking into account (7), we obtain

$$y(t) = b + \lambda B_\alpha^{-\beta}[y](t), \quad 0 < t < d.$$

Thus, if a function $y(t)$ is a solution to problem (6), (7), then it satisfies the following Volterra integral equation of the second kind

$$y(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t K(t,\tau) y(\tau) d\tau + b, \tag{8}$$

where $K(t, \tau) = (\ln t - \ln \tau)^{\alpha-1} \tau^{\beta-1}$.

We apply the method of normalized systems [26] to solve the integral equation (8). For this, we denote $L_1 = E, L_2 = \lambda B_\alpha^{-\alpha}$, where E is the identity operator. Then one can rewrite equation (8) in the form $(L_1 - L_2)\, y(t) = b$. Since $L_1 = E$, we have $L_1^{-1} = E$. Let $g_0 = b$. Following the technique used in [26], we consider the system

$$g_k(t) = \left(L_1^{-1} \cdot \lambda L_2\right)^k g_0, \quad k = 1, 2, ...$$

For $k=1$, we have

$$g_1(t)=\left(E\cdot\lambda B_\alpha^{-\beta}\right)g_0=\lambda B_\alpha^{-\beta}[b]=\frac{\lambda b}{\Gamma(\alpha)}\int\limits_0^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\tau^{\beta-1}d\tau=\left(\ln\frac{t}{\tau}=\xi\right)$$

$$=\frac{\lambda b}{\Gamma(\alpha)}\int\limits_0^\infty\xi^{\alpha-1}t^{\beta-1}e^{-(\beta-1)\xi}te^{\xi}d\xi=\frac{\lambda bt^\beta}{\Gamma(\alpha)}\int\limits_0^\infty\xi^{\alpha-1}e^{-\beta\xi}d\xi=(b\xi=s)$$

$$=\frac{\lambda bt^\beta}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^\infty s^{\alpha-1}e^{-s}ds=\frac{\lambda b}{\beta^\alpha}t^\beta.$$

Let $k=2$. Then we get

$$g_2(t)=\left(E\cdot\lambda B_\alpha^{-\beta}\right)g_1(t)=b\frac{\lambda^2}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\tau^{2\beta-1}d\tau=\left(\ln\frac{t}{\tau}=\xi\right)$$

$$=b\frac{\lambda^2t^{2\beta}}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^\infty\xi^{\alpha-1}e^{-2\beta\xi}d\xi=(2\beta\xi=s)=b\frac{\lambda^2t^{2\beta}}{2^\alpha\beta^\alpha\beta^{2\alpha}}.$$

Further, by the method of mathematical induction, one can prove that for any $k\geq 2$ the functions $g_k(t)$ can be represented in the form

$$g_k(t)=\left(E\cdot\lambda B_\alpha^{-\beta}\right)^k b=b\frac{\lambda^k}{(k!)^\alpha\beta^{k\alpha}}t^{k\beta}.$$

Lemma 4. *Let $0<\alpha\leq 1$ and $\beta>0$. Then problem (6), (7) has a unique solution and this solution is represented in the form*

$$y(t)=b\sum_{k=0}^\infty\frac{\lambda^k}{(k!)^\alpha\beta^{k\alpha}}t^{k\beta}. \tag{9}$$

Proof. Let us investigate the convergence of the series (9). To this end, we estimate the ratio $\frac{g_{k+1}(t)}{g_k(t)}$. We get

$$\left|\frac{g_{k+1}(t)}{g_k(t)}\right|=\left|\frac{\lambda^{k+1}t^{\beta(k+1)}}{\beta^{\alpha(k+1)}((k+1)!)^\alpha}\right|:\left|\frac{\lambda^k t^{\beta k}}{\beta^{\alpha k}(k!)^\alpha}\right|=\frac{|\lambda|\,t^\beta}{\beta^\alpha k}\xrightarrow[k\to\infty]{}0.$$

Hence, by the Dalamber theorem, the series (9) converges uniformly for $t\in[0,d]$ (in general, for $t\geq 0$). Since the functions $g_k(t)=\frac{\lambda^k}{\beta^{\alpha k}(k!)^\alpha}t^{\beta k}$, $k=1,2,...$, are continuous on $[0,+\infty)$, the sum of this series is continuous on $[0,d]$. Applying the operator $t^{-\beta}D^\alpha$ to the functions $g_k(t)$, and taking into account (4), we obtain

$$t^{-\beta}D^\alpha g_0(t)=0,$$

$$t^{-\beta}D^\alpha g_k(t)=t^{-\beta}D^\alpha\left[\frac{\lambda^k}{\beta^{\alpha k}(k!)^\alpha}t^{\beta k}\right]=\frac{\lambda^k}{\beta^{\alpha k-1}((k-1)!)^\alpha}t^{\beta(k-1)}=\lambda g_{k-1}(t),\quad k\geq 1.$$

Then, formally applying the operator $t^{-\beta}D^\alpha$ to series (9), we obtain

$$t^{-\beta}D^\alpha y(t)=\sum_{k=0}^\infty t^{-\beta}D^\alpha g_k(t)=\lambda\sum_{k=1}^\infty g_{k-1}(t)=\lambda\sum_{k=0}^\infty g_k(t)=\lambda y(t). \tag{10}$$

This implies the uniform convergence of the series $\sum\limits_{k=0}^{\infty} t^{-\beta} D^{\alpha} g_k(t)$ and the fulfilment of the condition $t^{-\beta} D^{\alpha} y(t) \in C[0, d]$. It follows from (10) that the function $y(t)$ in (9) satisfies the equation (6). It is obvious that $y(0) = b$, i.e. condition (7) is also met. This completes the proof of the lemma 4. □

Remark. The function (9) can be represented in the form $y(t) = L_{\alpha}\left(\frac{\lambda t^{\beta}}{\beta^{\alpha}}\right)$, where $L_{\alpha}(z)$ is defined as

$$L_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k!]^{\alpha}}, \quad \alpha > 0, \quad z \in R \tag{11}$$

and is called the Le Roy function [27, 28].

3. THE EXISTENCE AND UNIQUENESS OF A SOLUTION TO AN AUXILIARY PROBLEM

In this section, we study the following auxiliary problem

$$t^{-\beta} D^{\alpha} w(t, x) = \mu \Delta w(t, x), \tag{12}$$

$$w(0, x) = g(x), x \in \bar{\Pi}, \tag{13}$$

$$w(t, x) = 0, \quad 0 \le t \le T, \quad x \in \partial\Pi. \tag{14}$$

A function $w(t, x) \in C(\bar{\Omega}) \cap C^{\alpha,2,2}_{t,x_1,x_2}(\Omega)$ satisfying (12)–(14) in the classical sense is called a solution to problem (12)–(14).

We will seek the solution of problem (12)–(14) in the form $w(x, t) = z(x) \cdot T(t)$. Substituting this expression into equation (12) and boundary conditions (14), we obtain the spectral problem

$$-\Delta z(x) = \lambda z(x), \quad x \in \Pi, \quad z(x) = 0, \quad x \in \partial\Pi, \tag{15}$$

where λ is a spectral parameter.

The following statement is known (see, e.g., [29]).

Lemma 5. 1) *The eigenfunctions and the eigenvalues of problem* (15) *have the following form*

$$z_{k,m}(x_1, x_2) = X_k(x_1) Y_m(x_2) \equiv \sqrt{\frac{2}{p}} \sin\frac{k\pi}{p} x_1 \cdot \sqrt{\frac{2}{q}} \sin\frac{m\pi}{q} x_2, \quad k,\ m = 1, 2, ..., \tag{16}$$

and

$$\lambda_{k,m} = \nu_k + \sigma_m \equiv \left(\frac{k\pi}{p}\right)^2 + \left(\frac{m\pi}{q}\right)^2, \quad k, m = 1, 2, ...,$$

respectively.

2) *The functions in* (16) *form a complete orthonormal system in the space* $L_2(\Pi)$.

3.1. The Uniqueness of a Solution

Theorem 1. *Let* $\mu > 0$. *If a solution to problem (12)–(14) exists, then it is unique.*

Proof. Let us show that the problem under consideration has only the trivial solution. Assume, to the contrary, that there exist two solutions $w_1(t, x)$ and $w_2(t, x)$ to problem (12)–(14).

Let $\tilde{w}(t, x) = w_1(t, x) - w_2(t, x)$. It may be verified without difficulty that the function $\tilde{w}(t, x)$ satisfies the equation

$$t^{-\beta} D^{\alpha} \tilde{w}(t, x) = \mu \Delta \tilde{w}(t, x) \tag{17}$$

and conditions

$$\tilde{w}(0, x) = 0, \quad x \in \bar{\Pi}, \tag{18}$$

$$\tilde{w}(t,x)=0, \quad 0\le t\le T, \quad x\in\partial\Pi. \tag{19}$$

It remains to show that this problem has only the trivial solution. Let $\tilde{w}(t,x)$ be a solution to problem (17)–(19). We introduce the functions

$$\tilde{w}_{k,m}(t)=\left(\tilde{w}(t,x),\ z_{k,m}(x)\right)_{L_2(\Pi)}, \quad k,m=1,2,..., \tag{20}$$

where $(\xi,\eta)_{L_2(\Pi)}$ stands for the scalar product in the space $L_2(\Pi)$, i.e.

$$(\xi,\eta)_{L_2(\Pi)}=\iint\limits_{\Pi}\xi(x_1,\ x_2)\eta(x_1,\ x_2)dx_1dx_2.$$

Applying the operator $t^{-\beta}D^{\alpha}$ to both sides of (20) with respect to t, and taking into account (17)–(19), we get the following problem for finding the unknown functions $\tilde{w}_{k,m}(t)$:

$$\begin{cases} t^{-\beta}D^{\alpha}\tilde{w}_{k,m}(t)+\lambda_{ik}\mu\tilde{w}_{k,m}(t)=0, \quad t>0, \quad k,m=1,2,...,\\ \tilde{w}_{k,m}(0)=0. \end{cases} \tag{21}$$

The general solution to this problem (21) has the following form

$$\tilde{w}_{k,m}(t)=C_{k,m}\,L_{\alpha}\left(-\frac{\lambda_{ik}\mu}{\beta^{\alpha}}t^{\beta}\right), \quad k,m=1,2,...$$

Here $C_{k,m}$ are constant, $L_{\gamma}(z)$ is the Le Roy function defined in [27]. Further, substituting this solution into the initial condition in (21), we obtain $C_{k,m}=0$. This means that the problem under consideration has only the trivial solution $\tilde{w}_{k,m}(t)=0$.

Since $\tilde{w}_{k,m}(t)=0$, it follows from (20) that the function $\tilde{w}(t,x)$ is orthogonal to all functions of the system $z_{k,m}(x)$, which is complete in $L_2(\Pi)$. Hence, $\tilde{w}(t,x)=0$, i.e.

$$\tilde{w}(t,x)=0\Leftrightarrow w_1(t,x)=w_2(t,x).$$

This completes the proof of the uniqueness of the solution to problem (12)–(14). □

3.2. The Existence of a Solution To Problem I

Theorem 2. *Suppose that $\mu>0$ and a function $g(x)$ satisfies the conditions*

$$g(x_1,x_2)\in C^{r+s}_{x_1,x_2}(\bar{\Omega}), \quad r,\ s=0,1,2, \quad r+s\le 3,$$
$$g(x)=0, \quad x\in\partial\Pi.$$

Then there exists a solution to problem (12)–(14) and it is represented in the form

$$w(t,x)=\frac{2}{\sqrt{pq}}\sum_{k,m=1}^{\infty}g_{k,m}R_{\alpha}\left(-\frac{\mu\lambda_{k,m}}{\beta^{\alpha}}t^{\beta}\right)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2, \tag{22}$$

where $g_{k,m}$ are the Fourier coefficients of the expansion of $g(x)$ into the Fourier series over the system (16), i.e.

$$g_{k,m}=\left(g(x),\ z_{k,m}(x)\right)_{L_2(\Pi)}, \quad k,\ m=1,2,...,$$

the notation $C^{r+s}_{x_1,x_2}(\bar{\Omega})$ means that this set contains also mixed partial derivatives of order $r+s\le 3$.

Proof. Since the system of eigenfunctions $z_{k,m}(x)$, $k,\ m=1,2,...$ of problem (15) forms an orthonormal basis in $L_2(\Pi)$, a solution to problem (12)–(14) can be represented as the expansion into a series over this system:

$$w(t,x)=\sum_{k,m=1}^{\infty}w_{k,m}(t)z_{k,m}(x), \tag{23}$$

where $w_{k,m}(t),\ k,\ m = 1, 2, ...$ are unknown functions. Substituting the function (23) into (12) and (13), we obtain the Cauchy problem

$$\begin{cases} t^{-\beta} D^{\alpha} \tilde{w}_{k,m}(t) + \lambda_{ik} \mu \tilde{w}_{k,m}(t) = 0, \quad t > 0, \quad k,\ m = 1, 2, ..., \\ \tilde{w}_{k,m}(0) = g_{k,m}, \end{cases}$$

whose solution has the form

$$w_{k,m}(t) = g_{k,m} L_{\alpha} \left(-\frac{\mu \lambda_{k,m}}{\beta^{\alpha}} t^{\beta} \right), \quad k,\ m = 1, 2, ...,$$

where $L_{\alpha}(z)$ is the Le Roy function defined by (11).

Substituting the expressions for $w_{k,m}(t)$ into (23), we obtain the solution to the problem in the form of the series (22). Hence, the solution $w(x,t)$ to problem (12)–(14) can be represented in the form (22). It can be easily shown by direct calculations that the function $w(x,t)$ satisfies the equation (12) and the conditions (13) and (14). It remains to prove the validity of these operations.

Let us first give some relations that we will use in the proof of Theorem 2. Let $y(x_1, x_2)$ be a function satisfying the conditions of Theorem 2 and let $y_{k,m} = (y(x), z_{k,m}(x))_{L_2(\Pi)}, k, m = 1, 2, ...$ It is not difficult to obtain the following representations for the Fourier coefficients of this function:

$$y_{k,m} = \frac{pq}{\pi^2 km} y_{k,m}^{1,1} = -\frac{p^2 q}{\pi^2 k^2 m} y_{k,m}^{2,1} = -\frac{pq^2}{\pi^2 km^2} y_{k,m}^{1,2}, \tag{24}$$

where

$$y_{k,m}^{1,1} = \frac{2}{\sqrt{pq}} \int\limits_0^q \int\limits_0^p \frac{\partial^2 y(x_1,x_2)}{\partial x_1^1 \partial x_2^1} \cos \frac{k\pi}{p} x_1 \cos \frac{m\pi}{q} x_2 dx_1 dx_2,$$

$$y_{k,m}^{2,1} = \frac{2}{\sqrt{pq}} \int\limits_0^q \int\limits_0^p \frac{\partial^3 y(x_1,x_2)}{\partial x_1^2 \partial x_2^1} \sin \frac{k\pi}{p} x_1 \cos \frac{m\pi}{q} x_2 dx_1 dx_2,$$

$$y_{k,m}^{1,2} = \frac{2}{\sqrt{pq}} \int\limits_0^q \int\limits_0^p \frac{\partial^3 y(x_1,x_2)}{\partial x_1 \partial x_2^2} \cos \frac{k\pi}{p} x_1 \sin \frac{m\pi}{q} x_2 dx_1 dx_2.$$

Let us show: $w(x,t) \in C_{t,x_1,x_2}^{0,1,1}(\bar{\Omega})$. To this end, we calculate the partial derivatives

$$\frac{\partial w(t,x)}{\partial x_1} = \frac{2\pi}{p\sqrt{pq}} \sum_{k,m=1}^{\infty} k g_{k,m} R_{\alpha} \left(-\frac{\lambda_{k,m} \varepsilon}{\beta^{\alpha}} t^{\beta} \right) \cos \frac{k\pi}{p} x_1 \sin \frac{m\pi}{q} x_2 \tag{25}$$

and

$$\frac{\partial w(t,x)}{\partial x_2} = \frac{2\pi}{q\sqrt{pq}} \sum_{k,m=1}^{\infty} m g_{k,m} R_{\alpha} \left(-\frac{\lambda_{k,m} \varepsilon}{\beta^{\alpha}} t^{\beta} \right) \sin \frac{k\pi}{p} x_1 \cos \frac{m\pi}{q} x_2. \tag{26}$$

Further, using the estimate for the Le Roy function [27]

$$L_{\alpha}(-t) \le \frac{1}{1+t}, \quad t \ge 0, \quad 0 < \alpha < 1, \tag{27}$$

we get that the series (22), (25) and (26) are majorated by the following numerical series

$$\sum_{k=1}^{\infty} \sum_{i=1}^{2} (k+m) |g_{km}|.$$

Hence, taking into account (24) and applying the Cauchy–Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{\infty} \sum_{i=1}^{2} (k+m) |g_{km}| \le C \left(||g_{xxy}||_{L_2(\Pi)} + ||g_{xyy}||_{L_2(\Pi)} \right).$$

By the Weierstrass theorem, we get that the series (22), (25), and (26) converge absolutely and uniformly in $\bar{\Omega}$.

Let us now show that $w \in C^{\alpha,2,2}_{t,x_1,x_2}(\Omega)$. To this end, we calculate: $t^{-\beta}D^{\alpha}w(t,x)$, $\frac{\partial^2 w(t,x)}{\partial x_1^2}$ and $\frac{\partial^2 w(t,x)}{\partial x_2^2}$. So, we have

$$t^{-\beta}D^{\alpha}w(t,x) = -\frac{2\mu}{\sqrt{pq}}\sum_{k,m=1}^{\infty}\lambda_{km}g_{k,m}L_{\alpha}\left(-\frac{\mu\lambda_{k,m}}{\beta^{\alpha}}t^{\beta}\right)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2, \tag{28}$$

$$\frac{\partial^2 w(t,x)}{\partial x_1^2} = -\frac{2\pi^2}{p^2\sqrt{pq}}\sum_{k,m=1}^{\infty}k^2 g_{k,m}L_{\alpha}\left(-\frac{\lambda_{k,m}\varepsilon}{\beta^{\alpha}}t^{\beta}\right)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2, \tag{29}$$

$$\frac{\partial^2 w(t,x)}{\partial x_2^2} = \frac{2\pi^2}{q^2\sqrt{pq}}\sum_{k,m=1}^{\infty}m^2 g_{k,m}L_{\alpha}\left(-\frac{\lambda_{k,m}\varepsilon}{\beta^{\alpha}}t^{\beta}\right)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2. \tag{30}$$

Since $t \geq \varepsilon > 0$, by virtue of (27), the series (28)–(30) are majorated by the numerical series $\sum\limits_{k=1}^{\infty}\sum\limits_{m=1}^{\infty}|g_{km}|$. Further, taking into account (24) and using the Cauchy–Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{\infty}|g_{km}| \leq C||g_{xy}||_{L_2(\Pi)}.$$

It follows from the Weierstrass theorem that the series (28)–(30) converge absolutely and uniformly in Ω. Thus, if the condition of Theorem 2 is met, there exists a solution to problem (12)–(14) that is unique and is determined as the sum of the series (22). □

4. THE REDUCTION OF A SOLUTION TO THE NONLOCAL EQUATION TO THAT OF A LOCAL EQUATION

Let us consider equation (1). Replacing a point x by $S_j x$, $j = 1,2,3$, and taking into account the properties of the transformation S_j, we obtain the system of algebraic equations

$$t^{-\beta}D^{\alpha}U = A\Delta U, \tag{31}$$

where

$$U = \begin{pmatrix} u(t,x) \\ u(t,S_1x) \\ u(t,S_2x) \\ u(t,S_3x) \end{pmatrix}, \quad A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}.$$

Let J be the Jordan form of the matrix A and let Q be the transformation matrix such that the equality $A = Q \cdot J \cdot Q^{-1}$ holds. Since A is a symmetric matrix, the matrices Q and J have the forms

$$Q = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 \\ 0 & 0 & 0 & \varepsilon_4 \end{pmatrix}; \quad Q^{-1} = 4Q,$$

where ε_j, $j = 1,2,3,4$ are the eigenvalues of A and they are determined by the equalities

$$\varepsilon_1 = a_0 + a_1 + a_2 + a_3; \varepsilon_2 = a_0 + a_1 - a_2 - a_3; \varepsilon_3 = a_0 - a_1 + a_2 - a_3; \varepsilon_4 = a_0 - a_1 - a_2 + a_3.$$

In what follows, we will assume that $\varepsilon_j \neq 0$, $j = 1, 2, 3, 4$. The equation (31) can be rewritten as

$$t^{-\beta} D^{\alpha} U = \Delta \left(Q \cdot J \cdot Q^{-1} \right) U.$$

Hence,

$$t^{-\beta} D^{\alpha} Q^{-1} U = \Delta J \cdot Q^{-1} U. \tag{32}$$

Let $W = Q^{-1} U$, where

$$W = \begin{pmatrix} w_1(t,x) \\ w_2(t,x) \\ w_3(t,x) \\ w_4(t,x) \end{pmatrix}.$$

We can now represent equation (32) in the form

$$t^{-\beta} D^{\alpha} W = \Delta \left(J \cdot W \right) \Leftrightarrow t^{-\beta} D^{\alpha} W = \varepsilon \Delta W$$

or

$$t^{-\beta} D^{\alpha} w_j(t,x) = \varepsilon_j \Delta w_j(t,x), (t,x) \in \Omega, \quad j = 1, 2, 3, 4, \tag{33}$$

where

$$\begin{aligned} w_1(t,x) &= u(t,x) + u(t,S_1 x) + u(t,S_2 x) + u(t,S_3 x); \\ w_2(t,x) &= u(t,x) + u(t,S_1 x) - u(t,S_2 x) - u(t,S_3 x); \\ w_3(t,x) &= u(t,x) - u(t,S_1 x) + u(t,S_2 x) - u(t,S_3 x); \\ w_4(t,x) &= u(t,x) - u(t,S_1 x) - u(t,S_2 x) + u(t,S_3 x). \end{aligned} \tag{34}$$

Thus, we have proved the following statement.

Lemma 6. *If $\varepsilon_j, j = 1, 2, 3, 4$ are the eigenvalues of the matrix A, and a function $u(t,x)$ satisfies equation* (1), *then the functions $w_j(t,x)$, $j = 1, 2, 3, 4$ in* (34) *are the solutions to equations* (33).

If the functions $w_j(t,x), j = 1, 2, 3, 4$ are given in system (34), then, due to the fact that $\det Q^{-1} \neq 0$, from the system of equations $W = Q^{-1} U$ we get $U = QW$ or, what is the same,

$$\begin{aligned} u(t,x) &= \frac{1}{4} \left[w_1(t,x) + w_2(t,x) + w_3(t,x) + w_4(t,x) \right]; \\ u(t,S_1 x) &= \frac{1}{4} \left[w_1(t,x) + w_2(t,x) - w_3(t,x) - w_4(t,x) \right]; \\ u(t,S_2 x) &= \frac{1}{4} \left[w_1(t,x) - w_2(t,x) + w_3(t,x) - w_4(t,x) \right]; \\ u(t,S_3 x) &= \frac{1}{4} \left[w_1(t,x) - w_2(t,x) - w_3(t,x) + w_4(t,x) \right]. \end{aligned}$$

Lemma 7. *Let ε_j, $j = 1, 2, 3, 4$ be the eigenvalues of the matrix A, and let the functions $w_j(t,x), j = 1, 2, 3, 4$ are the solutions to equations* (33). *Then the function*

$$u(t,x) = \frac{1}{4} \left[w_1(t,x) + w_2(t,x) + w_3(t,x) + w_4(t,x) \right] \tag{35}$$

satisfies equation (1).

Proof. To prove this lemma, it is sufficient to show that the function $u(t,x)$ from (35) satisfies (1) or, what is the same, $t^{-\beta} D^{\alpha} U - \Delta A U = 0$. Let us apply the operators $t^{-\beta} D^{\alpha}$ and Δ to the equation $U = QW$. Then we have

$$t^{-\beta} D^{\alpha} U = t^{-\beta} D^{\alpha} Q W, \quad \Delta U = \Delta Q W.$$

Since

$$\Delta U = \Delta QW \Rightarrow \Delta AU = \Delta AQW \Leftrightarrow \Delta AU = Q \cdot J \cdot Q^{-1} Q \Delta W \Leftrightarrow$$
$$\Delta AU = Q\Delta\left(J \cdot W\right) \Rightarrow \Delta AU = Q\Delta\varepsilon W,$$

we have

$$t^{-\beta} D^{\alpha} U - \Delta AU = t^{-\beta} D^{\alpha}(QW) - \Delta\varepsilon(QW) = Q\left(t^{-\beta} D^{\alpha} - \varepsilon\Delta\right) W = Q \cdot 0 = 0$$

or $t^{-\beta} D^{\alpha} U - \Delta AU = 0$. Hence, we get, in particular, that equality (1) also holds. The lemma 7 is proved. □

5. THE INVESTIGATION OF PROBLEM D

Theorem 3. *Let $\varepsilon_j > 0$, $j = 1, 2, 3, 4$ and a function $\varphi(x)$ satisfies the conditions*

$$\varphi(x_1,\, x_2) \in C^{r+s}_{x_1,\, x_2}(\bar{\Omega}), \quad r,\, s = 0, 1, 2, r + s \leq 3, \quad \varphi(x) = 0, \quad x \in \partial\Pi.$$

Then there exists a solution to problem D, that is unique and is represented in the form

$$u(t,x) = \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k-1,2m-1} L_{\alpha}\left(-\varepsilon_1 \lambda_{2k-1,2m-1} \frac{t^{\beta}}{\beta^{\alpha}}\right) \sin\frac{(2k-1)\pi}{p} x_1 \cdot \sin\frac{(2m-1)\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k-1,2m} L_{\alpha}\left(-\varepsilon_2 \lambda_{2k-1,2m} \frac{t^{\beta}}{\beta^{\alpha}}\right) \sin\frac{(2k-1)\pi}{p} x_1 \cdot \sin\frac{2m\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k,2m-1} L_{\alpha}\left(-\varepsilon_3 \lambda_{2k,2m-1} \frac{t^{\beta}}{\beta^{\alpha}}\right) \sin\frac{2k\pi}{p} x_1 \cdot \sin\frac{(2m-1)\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k,2m} L_{\alpha}\left(-\varepsilon_4 \lambda_{2k,2m} \frac{t^{\beta}}{\beta^{\alpha}}\right) \sin\frac{2k\pi}{p} x_1 \cdot \sin\frac{2m\pi}{q} x_2, \tag{36}$$

where $\varphi_{k,m}$ are the Fourier coefficients of the expansion of the function $\varphi(x)$ into the Fourier series over the system (16), i.e. $\varphi_{k,m} = \left(\varphi(x), z_{k,m}(x)\right)_{L_2(\Pi)}$, $k,\, m = 1, 2, ...$

Proof. Let us consider the following problems for functions $w_j(t,x)$, $j = 1, 2, 3, 4$:

$$t^{-\beta} D^{\alpha} w_j(t,x) = \varepsilon_j \Delta w_j(t,x), \quad (t,x) \in \Omega, \tag{37}$$

$$w_j(0,x) = \varphi_j(x), \quad x \in \bar{\Pi}, \tag{38}$$

$$w_j(t,x) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Pi. \tag{39}$$

Here the functions $\varphi_j(x)$, $j = 1, 2, 3, 4$ are defined as

$$\varphi_1(x) = \varphi(x) + \varphi(S_1 x) + \varphi(S_2 x) + \varphi(S_3 x);$$
$$\varphi_2(x) = \varphi(x) + \varphi(S_1 x) - \varphi(S_2 x) - \varphi(S_3 x);$$
$$\varphi_3(x) = \varphi(x) - \varphi(S_1 x) + \varphi(S_2 x) - \varphi(S_3 x);$$
$$\varphi_4(x) = \varphi(x) - \varphi(S_1 x) - \varphi(S_2 x) + \varphi(S_3 x). \tag{40}$$

If the function $\varphi(x)$ satisfies the conditions of Theorem 3, then so do the functions $\varphi_j(x)$ also satisfy the conditions of the Theorem 3. Hence, according to the Theorems 1 and 2, there exist solutions to problem (37)–(39), that are unique, belong to the class $C(\bar{\Omega}) \cap C^{\alpha,2,2}_{t,x_1,x_2}(\Omega)$ and are represented in the form

$$w_j(t,x) = \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{j,k,m} L_{\alpha}\left(-\frac{\varepsilon_j \lambda_{k,m}}{\beta^{\alpha}} t^{\beta}\right) \sin\frac{k\pi}{p} x_1 \sin\frac{m\pi}{q} x_2, \quad j = 1, 2, 3, 4. \tag{41}$$

Using these functions (41), we construct the function $u(t,x)$ by formula (35) and show that $u(t,x)$ is a solution to Problem D. By Lemma 7, the function $u(t,x)$ from (35) satisfies equation (1). Let us check the fulfilment of conditions (2) and (3). For $t=0$, we have

$$u(0,x)=\frac{1}{4}\left[w_1(0,x)+w_2(0,x)+w_3(0,x)+w_4(0,x)\right]=\frac{1}{4}\left[\varphi_1(x)+\varphi_2(x)+\varphi_3(x)+\varphi_4(x)\right]$$
$$=\frac{1}{4}\left[\varphi(x)+\varphi(S_1x)+\varphi(S_2x)+\varphi(S_3x)\right]+\frac{1}{4}\left[\varphi(x)+\varphi(S_1x)-\varphi(S_2x)-\varphi(S_3x)\right]$$
$$+\frac{1}{4}\left[\varphi(x)-\varphi(S_1x)+\varphi(S_2x)-\varphi(S_3x)\right]+\frac{1}{4}\left[\varphi(x)-\varphi(S_1x)-\varphi(S_2x)+\varphi(S_3x)\right]=\varphi(x).$$

Similarly, for the points $t\in[0,T]$, $x\in\partial\Pi$ we get from (33) that

$$u(t,x)=\frac{1}{4}\left[w_1(t,x)+w_2(t,x)+w_3(t,x)+w_4(t,x)\right]=0,\quad t\in[0,T],\quad x\in\partial\Pi.$$

Hence, the function (35) is a solution to Problem D. Let us find its representation. The following equalities hold for the function $\varphi(x_1,\,x_2)$:

$$\int_0^q\int_0^p \varphi(p-x_1,x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2$$
$$=-(-1)^k\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2,$$
$$\int_0^q\int_0^p \varphi(x_1,q-x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2$$
$$=-(-1)^m\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2,$$
$$\int_0^q\int_0^p \varphi(p-x_1,q-x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2$$
$$-(-1)^{k+m}\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2.$$

Then, using the representation (40) for the Fourier coefficients of the functions $\varphi_j(x)$, we obtain

$$\varphi_{1,k,m}=\tau_1\cdot\frac{2}{\sqrt{pq}}\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{k\pi}{p}x_1\cdot\sin\frac{m\pi}{q}x_2dx_1dx_2$$
$$=4\frac{2}{\sqrt{pq}}\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{(2k-1)\pi}{p}x_1\cdot\sin\frac{(2m-1)\pi}{q}x_2dx_1dx_2,$$
$$\varphi_{2,k,m}=\tau_2\cdot\frac{2}{\sqrt{pq}}\int_0^q\int_0^p \varphi(x_1,x_2)\sin\frac{(2k-1)\pi}{p}x_1\cdot\sin\frac{2m\pi}{q}x_2dx_1dx_2$$
$$=4\frac{2}{\sqrt{pq}}\int_0^q\int_0^p \varphi(x_1,\,x_2)\sin\frac{(2k-1)\pi}{p}x_1\cdot\sin\frac{2m\pi}{q}x_2dx_1dx_2,$$

$$\varphi_{3,k,m} = \tau_3 \cdot \frac{2}{\sqrt{pq}} \int_0^q \int_0^p \varphi(x_1, x_2) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{(2m-1)\pi}{q} x_2 dx_1 dx_2$$

$$= 4 \frac{2}{\sqrt{pq}} \int_0^q \int_0^p \varphi(x_1, x_2) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{(2m-1)\pi}{q} x_2 dx_1 dx_2,$$

$$\varphi_{4,k,m} = \tau_4 \cdot \frac{2}{\sqrt{pq}} \int_0^q \int_0^p \varphi(x_1, x_2) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{2m\pi}{q} x_2 dx_1 dx_2$$

$$= 4 \frac{2}{\sqrt{pq}} \int_0^q \int_0^p \varphi(x_1, x_2) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{2m\pi}{q} x_2 dx_1 dx_2,$$

where

$$\tau_1 = \left(1 + (-1)^{k+1} + (-1)^{m+1} + (-1)^{k+m+2}\right), \quad \tau_2 = \left(1 + (-1)^{k+1} - (-1)^{m+1} - (-1)^{k+m+2}\right),$$
$$\tau_3 = \left(1 - (-1)^{k+1} + (-1)^{m+1} - (-1)^{k+m+2}\right), \quad \tau_4 = \left(1 - (-1)^{k+1} - (-1)^{m+1} + (-1)^{k+m+2}\right).$$

Thus, for the solution to Problem D we get

$$u(t, x) = \frac{1}{4} \left[w_1(t, x) + w_2(t, x) + w_3(t, x) + w_4(t, x)\right]$$
$$= \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k-1,2m-1} L_\alpha \left(-\varepsilon_1 \lambda_{2k-1,2m-1} \frac{t^\beta}{\beta^\alpha}\right) \sin \frac{(2k-1)\pi}{p} x_1 \cdot \sin \frac{(2m-1)\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k-1,2m} L_\alpha \left(-\varepsilon_2 \lambda_{2k-1,2m} \frac{t^\beta}{\beta^\alpha}\right) \sin \frac{(2k-1)\pi}{p} x_1 \cdot \sin \frac{2m\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k,2m-1} L_\alpha \left(-\varepsilon_3 \lambda_{2k,2m-1} \frac{t^\beta}{\beta^\alpha}\right) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{(2m-1)\pi}{q} x_2$$
$$+ \frac{2}{\sqrt{pq}} \sum_{k,m=1}^{\infty} \varphi_{2k,2m} L_\alpha \left(-\varepsilon_4 \lambda_{2k,2m} \frac{t^\beta}{\beta^\alpha}\right) \sin \frac{2k\pi}{p} x_1 \cdot \sin \frac{2m\pi}{q} x_2,$$

i.e. the representation (36) is valid. The theorem 3 is proved. □

6. CONCLUSIONS

In this paper, the concept of a nonlocal Laplace operator, which is a generalization of the classical Laplace operator, has been introduced. Another generalization of the classical parabolic equation - a two-dimensional parabolic equation with involution is also introduced for the first time and using a spectral method in a three-dimensional parallelepiped, the solvability of an inverse problem was also studied. Using matrix theory, a method of reducing this problem for a parabolic equation with involution to the well-known problem of a parabolic equation without involution is proposed.

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