

Internal Boundary Layer for Integral-Differential Equations with Zero Spectrum of the Limit Operator and Rapidly Changing Kernel

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Abstract

In this paper we consider the Cauchy problem for systems of integral - differential equations with zero points of the spectrum of the limit operator. The presence of singularities in the integral term with a rapidly decreasing kernel generates in a solution of the original problem of essentially special singularities by a small parameter, which describes the internal boundary layer. To establish mathematical theory, criterion formulation of correctness of mathematical description of the boundary layer and to develop a regular theory for singularly perturbed problems we use the regularization method of S.A.Lomov. Normal and unique solvability of iterative tasks is proved, the asymptotic convergence of formal solutions is proved.

Keywords: boundary layer, asymptotical solution, perturbation, regularization, stability spectrum, limiting operator, iterative problems

1 Introduction

Mathematical theory of the boundary layer is associated with correct description singular dependence of solutions of the corresponding problem on a small parameter, i.e. with structure of solutions by a small parameter. In turn, description singular dependence of solutions on a small parameter is related

to properties of a spectrum of the limit operator or properties of roots of the characteristic equations, and therefore, in each problem it is necessary to allocate basis of singularities. Spectrum of the operator, as we know, can be either discrete or continuous. Hence discrete or continuum description singular dependence on the corresponding small parameter appears, that requires to distinguish discrete and continuum boundary layer [14].

We study mathematical theory of the boundary layer for linear singularly perturbed integral - differential system when the area, in which we study the problem - in this case it is a segment $[0, T]$, - is compact. Problem with identity irreversibility of the limit operator is considered in [18], where the Cauchy problem for systems of ordinary differential equations with weak nonlinearity was studied, limit system for which, unlike the present case, is homogeneous. Moreover, non-zero points of the spectrum could be only in the left half plane. For solutions the asymptotic of the boundary layer type was established. Singularly perturbed systems with irreversible limit operator in the case, when the eigenvalues are purely imaginary, were considered in [1,2,4,7,19,20,21], by the regularization method of S.A.Lomov [14].

At the end of the last century articles devoted to the study of singularly perturbed integral-differential systems with rapidly changing kernels began to appear [3,5,6,8,9,10,11,12,13,14,16,17]. Integral operators with rapidly decreasing kernel create in the structure of solutions of the original problem a new type of singularities by a small parameter, which describes the internal boundary layer. In this paper we consider an integral-differential system with zero values of the spectrum at discrete points and spectral singularities of the kernel of the integral operator.

Consider the following problem:

$$L_\varepsilon y(t, \varepsilon) \equiv \varepsilon \frac{dy}{dt} - A(t)y + \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \mu(x) dx} K(t, s)y(s, \varepsilon) ds = h(t),$$

$$y(0, \varepsilon) = y^0, \quad t \in [0, T] \quad (1)$$

where $y(t, \varepsilon) = \{y_1, \dots, y_n\}$, $A(t)$, $K(t, s)$ are matrix-valued functions of $(n \times n)$ dimension, $h(t) = \{h_1, \dots, h_n\}$ is a known vector-valued function, $y^0 \in C^n$ is a constant vector, $\mu(t) \in C^\infty[0, T]$ is a scalar function, $\varepsilon > 0$ is a small parameter. It is required to establish regularized asymptotics of solutions as $\varepsilon \rightarrow +0$.

In order to obtain asymptotic representations for the function $y(t, \varepsilon)$ in the form of series in powers of ε we require the following conditions:

- i) $A(t) \in C^\infty([0, T], C^{n^2})$, $h(t) \in C^\infty([0, T], C^n)$,
 $K(t, s) \in C^\infty(0 \leq s \leq t \leq T, C^{n^2})$;

for any $t \in [0, T]$ the spectrum $\{\lambda_j(t)\}$ of the operator $A(t)$ satisfies the conditions:

- ii) $\lambda_j(t) \neq 0, j = \overline{1, p}, \lambda_i(t) \equiv 0, i = \overline{p+1, n}, p < n;$
- iii) $\lambda_m(t) \equiv \mu(t) = -tk(t), k(t) \neq 0, k(t) \in C^\infty([0, T], R^1), \lambda_m(t) \neq \lambda_j(t), j = \overline{1, n}, \forall t \in [0, T].$

Here the scalar function $\mu(t)$ called spectral value of the kernel of the integral operator, vanishes at the point $t = 0$. It induces in a solution of the problem (1) additional rapidly changing components, i.e. essentially singularity.

The main difficulty in solving problems with degenerate limit operator is the fact that the degenerate system

$$0 = A(t)\bar{y} + h(t) \tag{2}$$

does not have any solutions, or has countless solutions. Therefore, it is not clear in advance to what solution of the system (2) the true solution $y(t, \varepsilon)$ of the problem (1) approaches (at $\varepsilon \rightarrow +0$). This leads to the fact that we cannot say from the outset, what restrictions should be imposed on the domain of the function $h(t)$, such that a statement was correct. Naturally, this difficulty would be overcome, if we could find by some method a limit solution of the problem (1). In 1976, this difficulty was overcome relatively simple by I.S.Lomov [15]. As we know, in this case the system (2) is solvable if the condition $h \perp Ker A^*$ holds for each $t \in [0, T]$. It should be noted that if $h(t)$ does not satisfy these conditions of orthogonality, then the solution of (1) under the natural assumption $Re\lambda \leq 0$ although it exists, but it will increase indefinitely at $\varepsilon \rightarrow +0$.

We assume that the degenerate system (2) has a solution. In this case the right hand side $h(t)$ of this system should be orthogonal to the kernel of the operator $A^*(t)$, i.e.

$$(h(t), d_i(t)) \equiv 0, \quad i = \overline{p+1, n}, \quad \forall t \in [0, T] \tag{3}$$

where $d_i(t)$ are eigenvectors of the matrix $A^*(t)$, which correspond to the zero eigenvalues $\lambda_i(t) \equiv 0$, and form a basis of the kernel of the operator $A^*(t)$.

We also assume that the matrix $A(t)$ is the operator of a simple structure. In this case, there is a complete system of eigenvectors $\{c_i(t)\}$ of the matrix $A(t)$, and zero eigenvalues $\{\lambda_i(t)\}$ of $n - p$ multiplicity corresponds to $n - p$ eigenvectors $c_i(t), i = \overline{p+1, n}$. Degenerate system (2) will have a set of solutions

$$\bar{y}(t, \alpha_{p+1}, \dots, \alpha_n) = \alpha_{p+1}(t)c_{p+1}(t) + \dots + \alpha_n(t)c_n(t) + \tilde{y}_0(t) \tag{4}$$

depending on $n - p$ arbitrary scalar function $\alpha_i(t)$, defined on the segment $[0, T]$. Here $\tilde{y}_0(t)$, is a partial solution of the system (2), corresponding to the right-hand side $h(t)$. Consider the following system of equations ($i = \overline{p+1, n}$)

$$\left(-\frac{d\bar{y}(t, \alpha_{p+1}(t), \dots, \alpha_n(t))}{dt}, d_i(t) \right) = 0 \tag{5}$$

where $\bar{y}(t, \alpha_{p+1}, \dots, \alpha_n)$ is a set (4). Set (5) is a system of ordinary differential equations ($n - p$ order) relatively to arbitrary functions $\alpha_i(t)$, $i = \overline{p+1, n}$. We put for (5) initial conditions:

$$\alpha_i(0) = \sum_{k=1}^n c_{ik}^{(-1)}(0)(y_k^0 - \tilde{y}_{0k}(0)), \quad i = \overline{p+1, n} \tag{6}$$

where by $c_{ik}^{(-1)}(t)$ we denote elements of the matrix $C^{-1}(t)$, inverse to the matrix $C(t)$, whose columns are the eigenvectors $c_i(t)$, y_k^0 and $\tilde{y}_{0k}(0)$ are components of the vectors y^0 and $\tilde{y}_0(0)$, $i, k = \overline{1, n}$, respectively. We show that if the problem (5) - (6) has a solution on the interval $[0, T]$, then the function (4), is a limit solution of the original problem.

It is important to note that to determine a limit solution it is necessary to solve the system (5) - (6). As mentioned above, such a solution exists. So, let

$$\bar{y} = \varphi(t) \equiv \alpha_{p+1}(t)c_{p+1}(t) + \dots + \alpha_n(t)c_n(t) + \tilde{y}_0(t) \tag{4'}$$

be a limit solution of the problem (1), where functions $\alpha_{p+1}(t), \dots, \alpha_n(t)$ satisfy the problem (5) - (6).

2 Regularization of the problem

We introduce regularization variables by nonzero points on the spectrum of the operator $A(t)$:

$$\tau_j = \frac{1}{\varepsilon} \int_0^t \lambda_j(x) dx \equiv \varphi_j(t, \varepsilon), \quad j = \overline{1, p}.$$

Moreover, we introduce additional variables, generated by integral term of the system (1):

$$\tau_m = \frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx \equiv \varphi_m(t, \varepsilon), \quad \sigma = e^{\frac{\varphi_m(t)}{\varepsilon}} \int_0^t e^{-\frac{\varphi_m(x)}{\varepsilon}} dx \equiv \psi(t, \varepsilon).$$

Instead of unknown function $y(t, \varepsilon)$ of the problem (1) we consider a function $\tilde{y}(t, \tau, \sigma, \varepsilon)$ with a large number of variables $\tilde{y}(t, \tau, \sigma, \varepsilon)|_{\tau=\varphi(t)/\varepsilon, \sigma=\psi(t)/\varepsilon} \equiv y(t, \varepsilon)$, where $\tau = (\tau_1, \tau_2, \dots, \tau_p, \tau_m)$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_p, \varphi_m)$.

Then for the function $\tilde{y}(t, \tau, \sigma, \varepsilon)$ it is naturally to consider the following problem:

$$\begin{aligned} \tilde{L}_\varepsilon \tilde{y}(t, \tau, \sigma, \varepsilon) &\equiv \varepsilon \frac{\partial \tilde{y}}{\partial t} + D\lambda \tilde{y} + \lambda_m(t) \frac{\partial \tilde{y}}{\partial \tau_m} + [\lambda_m(t)\sigma + \varepsilon] \frac{\partial \tilde{y}}{\partial \sigma} - A(t)\tilde{y} \\ &- \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \mu(x) dx} K(t, s)\tilde{y}(s, \varphi(s, \varepsilon), \psi(s, \varepsilon), \varepsilon) ds = h(t), \quad \tilde{y}(0, 0, 0, \varepsilon) = y^0 \end{aligned} \tag{7}$$

where $D_\lambda \equiv \sum_{j=1}^p \lambda_j(t) \frac{\partial}{\partial \tau_j} - A(t)$.

However the problem (7) is not yet "extended" in relation to the original problem (1), since regularization of an integral member is not made:

$$Iy = \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \mu(x) dx} K(t, s) \tilde{y}(s, \varphi(s, \varepsilon), \psi(s, \varepsilon)) ds. \tag{8}$$

To regularize the integral operator (8) we should describe the class M_ε , asymptotically invariant (as $\varepsilon \rightarrow +0$) to the operator I (see [14])

First we introduce a class U described as follows:

Definition 1 We say that the function $y(t, \tau, \sigma) = \{y_1, \dots, y_n\}$ belongs to the class U , if it can be represented as the sum:

$$y(t, \tau, \sigma) = y_0(t) + \sum_{j=1}^p y_j(t) e^{\tau_j} + y_m(t) e^{\tau_m} + y_{m+1}(t) \sigma \tag{9}$$

with coefficients $y_i(t), y_m(t), y_{m+1}(t) \in C^\infty[0, T], \quad i = \overline{0, p}$.

As a class M_ε we take a restriction of the class U when $\tau = \varphi(t)/\varepsilon, \sigma = \psi(t, \varepsilon)$. Let us prove, that $U|_{\tau=\varphi(t)/\varepsilon, \sigma=\psi(t, \varepsilon)}$ is invariant with respect to the integral operator I .

Theorem 1.1 Let conditions (i) - (iii) hold. Then the class $M_\varepsilon = U|_{\tau=\varphi(t)/\varepsilon, \sigma=\psi(t, \varepsilon)}$ is asymptotical invariant with respect to the integral operator I .

Proof Putting (9) into (8), we have

$$\begin{aligned} I_0(t, \varepsilon) &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} K(t, s) y_0(s) ds; \\ I_j(t, \varepsilon) &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t e^{\frac{1}{\varepsilon} \int_0^s [\lambda_j(x) - \lambda_m(x)] dx} K(t, s) y_j(s) ds, \quad j = \overline{1, p}; \\ I_m(t, \varepsilon) &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t K(t, s) y_m(s) ds; \\ I_{m+1}(t, s) &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t K(t, s) y_{m+1}(s) \left(\int_0^s e^{-\frac{1}{\varepsilon} \int_0^\tau \lambda_m(x) dx} d\tau \right) ds. \end{aligned}$$

We have to show that the integrals, containing exponents, are expanded into asymptotic series by powers of ε (as $\varepsilon \rightarrow +0$). Applying operation of integration by parts, we get:

$$\begin{aligned} I_0(t, \varepsilon) &\equiv I_0(t, \varphi(t, \varepsilon), \psi(t, \varepsilon), \varepsilon) \\ &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} \{ [K(t, s) - K_0(t, 0)] + K_0(t, 0) \} ds \end{aligned}$$

$$\begin{aligned}
 &= \psi(t, \varepsilon)K_0(t, 0) + e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} K_0^1(t, s) ds \\
 &= K_0(t, 0)\psi(t, \varepsilon) + \varepsilon e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t \frac{K_0^1(t, s)}{\mu(s)} d \left(e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} \right) \\
 &= K_0(t, 0)\psi(t, \varepsilon) + \varepsilon e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \left[\frac{K_0^1(t, s)}{\mu(s)} e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} \right]_{s=0}^{s=t} \\
 &\quad - \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} \frac{\partial}{\partial s} \left(\frac{K_0^1(t, s)}{\mu(s)} \right) ds = K_0(t, s)\psi(t, \varepsilon) + \varepsilon \frac{K_0^1(t, t)}{\mu(t)} e^{\varphi_m(t, \varepsilon)} \\
 &\quad - \varepsilon \frac{K_0^1(t, 0)}{\mu(0)} - \varepsilon e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} \frac{\partial}{\partial s} \left(\frac{K_0^1(t, s)}{\mu(s)} \right) ds
 \end{aligned}$$

where $K_0(t, s) = K(t, s)y_0(s)$, $K_0^1(t, s) = \frac{K_0(t, s) - K_0(t, 0)}{s}$.

So, after a single integration by parts, terms out of the integral are distinguished, which have the form of the sums (9) when $\tau = \varphi$, $\sigma = \psi$ and the integral term is again an integral type $I_0(t, \varepsilon)$. Repeated integration by parts leads to the formal series:

$$\begin{aligned}
 I_0(t, \tau, \sigma, \varepsilon) &= K_0(t, s)\sigma + \varepsilon \left[\frac{K_0^1(t, t)}{\mu(t)} - \frac{K_0^1(t, 0)}{\mu(0)} e^{\tau_m} \right] \\
 &\quad + \sum_{k=2}^{\infty} \varepsilon^k [\nu_k(t) + \nu_k(t)\sigma + \nu_{km}(t)e^{\tau_m}] \tag{*}
 \end{aligned}$$

where $\tau = \frac{\varphi(t)}{\varepsilon}$, $\sigma = \psi(t, \varepsilon)$.

Regularization of integrals $I_j(t, \varepsilon)$, $j = \overline{1, p}$ is made as in [3]:

$$\begin{aligned}
 I_j(t, \tau, \sigma, \varepsilon) &= \varepsilon \left[\frac{K_j(t, t)}{\lambda_j(t) - \lambda_m(t)} e^{\tau_j} - \frac{K_j(t, 0)}{\lambda_j(0) - \lambda_m(0)} e^{\tau_m} \right] \\
 &\quad + \sum_{k=2}^{\infty} \varepsilon^k [v_{kj}(t)e^{\tau_j} + v_{km}(t)e^{\tau_m}]
 \end{aligned}$$

where $K_j(t, s) \equiv K(t, s)y_j(s)$, $j = \overline{1, p}$.

Integral $I_m(t, \varepsilon)$ is already regularized, i.e.

$$I_m(t, \varepsilon) \equiv I_0(t, \tau, \sigma, \varepsilon) = e^{\tau_m} \int_0^t K_m(t, s) ds$$

where $K_m(t, s) \equiv K(t, s)y_m(s)$.

We write the integral $I_{m+1}(t, \varepsilon)$ in the following form:

$$\begin{aligned}
 I_{m+1}(t, \varepsilon) &\equiv I_{m+1}(t, \varphi(t, \varepsilon), \psi(t, \varepsilon), \varepsilon) \\
 &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t \left(\int_0^s e^{-\frac{1}{\varepsilon} \int_0^\tau \lambda_m(x) dx} d\tau \right) d \left(\widehat{K}_{m+1}(t, s) \right) \\
 &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \left(\widehat{K}_{m+1}(t, s) \int_0^s e^{-\frac{1}{\varepsilon} \int_0^\tau \lambda_m(x) dx} d\tau \right) \Big|_{s=0}^{s=t} \\
 &\quad - e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t \widehat{K}_{m+1}(t, s) e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} ds \\
 &= e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \widehat{K}_{m+1}(t, t) \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} ds \\
 &\quad - e^{-\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t \widehat{K}_{m+1}(t, s) e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} ds \\
 &= \widehat{K}_{m+1}(t, t) \psi(t, \varepsilon) - e^{-\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \int_0^t \widehat{K}_{m+1}(t, s) e^{-\frac{1}{\varepsilon} \int_0^s \lambda_m(x) dx} ds
 \end{aligned}$$

where $K_{m+1}(t, s) = K(t, s)y_{m+1}(s)$, $\widehat{K}_{m+1}(t, s) = \int_0^s K_{m+1}(t, s) ds$.

Making the operation of integration by parts we get integral of the type $I_0(t, \varepsilon)$. Due to the regularization of the integral term $I_0(t, \varepsilon)$. we obtain the following expansion for the integral $I_{m+1}(t, \varepsilon)$:

$$\begin{aligned}
 I_{m+1}(t, \varepsilon) &\equiv I_{m+1}(t, \tau, \sigma, \varepsilon) = \widehat{K}_{m+1}^1(t, t) \sigma - \varepsilon \frac{\widehat{K}_{m+1}^1(t, t)}{\mu(t)} \\
 &\quad + \varepsilon \frac{\widehat{K}_{m+1}^1(t, 0)}{\mu(0)} e^{\tau m} - \widehat{K}_{m+1}(t, 0) \sigma + \sum_{k=2}^{\infty} \varepsilon^k [v_k(t) + v_{km}(t) e^{\tau m} + v_{k,m+1}(t) \sigma] \\
 &\quad \text{where } \widehat{K}_{m+1}^1(t, s) = \frac{\widehat{K}_{m+1}(t, s) - \widehat{K}_{m+1}(t, 0)}{s}.
 \end{aligned}$$

Lets show, for example, that the series (*) converges asymptotically as $\varepsilon \rightarrow +0$. Its partial sum

$$\begin{aligned}
 S_N(t, \varepsilon) &= \sum_{m=0}^{N-1} (-1)^m \varepsilon^{m+1} \left[\left(I_j^m(K_j(t, s)y_j^{(k)}(s)) \right)_{s=t} e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(x) dx} \right. \\
 &\quad \left. - \left(I_j^m(K_j(t, s)y_j^{(k)}(s)) \right)_{s=0} e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \right]
 \end{aligned}$$

where $I_j^0 = \frac{1}{\lambda_j(s)}$, $I_j^{m-1} = \frac{1}{\lambda_j(s)} \frac{\partial}{\partial s} I_j^m$, $m \geq 1$, $j = 1, 2, \dots, p, m$, satisfies equality:

$$I_j(t, \varepsilon) = S_N(t, \varepsilon) + (-1)^N \varepsilon^N \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \lambda_j(x) dx} \frac{\partial}{\partial s} \left(I_j^{N-1}(K_j(t, s)y_j(s)) \right) ds,$$

$$j = 1, 2, \dots, p, m.$$

Integrating by parts the integral from the right-hand side of the equality; we have

$$\begin{aligned} I_j(t, \varepsilon) - S_N(t, \varepsilon) &= (-1)^N \varepsilon^{N+1} \int_0^t \left(I_j^N(K_j(t, s)y_j(s)) \right) d \left(e^{\frac{1}{\varepsilon} \int_s^t \lambda_j(x) dx} \right) \\ &= (-1)^N \varepsilon^N \left\{ \left[\left(I_j^{N-1}(K_j(t, s)y_j(s)) \right)_{s=t} e^{\frac{1}{\varepsilon} \int_0^t \lambda_j(x) dx} \right. \right. \\ &\quad \left. \left. - \left(I_j^N(K_j(t, s)y_j(s)) \right)_{s=0} e^{\frac{1}{\varepsilon} \int_0^t \lambda_m(x) dx} \right] \right. \\ &\quad \left. - \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \lambda_j(x) dx} \frac{\partial}{\partial s} \left(I_j^N(K_j(t, s)y_j(s)) \right) ds \right\}, \quad j = 1, 2, \dots, p. \end{aligned}$$

Due to infinitely differentiability of functions $K_j(t, s)$ on $0 \leq s \leq t \leq T$ and $\lambda_j(t)$, $j = 1, 2, \dots, p, m$, on $[0, T]$, and due to the uniform boundedness of the exponents, functions standing in the outer brace of the last equality are uniformly bounded for $(t, \varepsilon) \in [0, 1] \times Re\tau_j \leq 0$, $j = 1, 2, \dots, p, m$. Consequently,

$$\|J_j(t, \varepsilon) - S_N(t, \varepsilon)\|_{C[0, T]} \leq C_N \varepsilon^{N+1}$$

where $C_N > 0$ is constant, which does not depend on ε as $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ infinitesimal). It means that the series (9) converges to the integral $I_j(t, \varepsilon)$ asymptotically as $\varepsilon \rightarrow +0$. But the the image $Iy(t, \varepsilon)$ is expanded into asymptotical series:

$$\begin{aligned} Iy(t, \tau, \sigma) &= \sum_{m=0}^{\infty} \varepsilon^{m+1} \left\{ \left[\left(I_j^m(K_j(t, s)y_j(s)) \right)_{s=t} e^{\tau_j} \right. \right. \\ &\quad \left. \left. - \left(I_j^m(G(t, s)y_j(s)) \right)_{s=0} e^{\tau_m} + \int_0^t K_j(t, s)y_j(s) ds \right] \right\} \end{aligned}$$

where $\tau = \psi(t)/\varepsilon$, $\sigma = \psi(t, \varepsilon)$. This proves that the class M_ε is asymptotically invariant with respect to the integral operator I . Theorem 1.1 is proved.

Now we construct an extension of the operator I .

Let the series

$$\tilde{y}(t, \tau, \sigma, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t, \tau, \sigma) \tag{10}$$

with coefficients $y_k(t, \tau, \sigma) \in C^\infty([0, T], C^n)$ be given.

For arbitrary element (9) of the space U we can write that

$$Iy(t, \tau, \sigma) = R_0y(t, \tau, \sigma) + \sum_{\nu=0}^{\infty} (R_{\nu+1}y(t, \tau, \sigma)) \cdot \varepsilon^{\nu+1}$$

where by R_ν we denote operators of ν order, defined by the formulas:

$$\begin{aligned} R_0y(t, \tau, \sigma) &= e^{\tau m} \int_0^t K(t, s)y_m(s)ds, \\ R_{\nu+1}y(t, \tau, \sigma) &= (-1)^\nu \cdot [(I_0^\nu (K(t, s)y_0(s)))_{s=t} \\ &\quad - (I_0^\nu (K(t, s)y_0(s)))_{s=0} \cdot e^{\tau m} + (I_j^\nu (K(t, s)y_j(s)))_{s=t} \cdot e^{\tau j} \\ &\quad - (I_j^\nu (K(t, s)y_j(s)))_{s=0} \cdot e^{\tau m}], \quad (\varepsilon^{-1}\varphi(t), \nu \geq 1). \end{aligned} \tag{11}$$

Due to these formulas, result of substitution of the series (10) into the integral operator (8) can be rewritten as follows:

$$\tilde{I}\tilde{y} = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=1, r-s \geq 0}^r R_{r-s}y_s(t, \tau, \sigma) \Big|_{\tau=\varepsilon^{-1}\varphi(t), \sigma=\psi(t, \varepsilon)}.$$

Definition 2 As a formal extension of the operator I we call the operator:

$$\begin{aligned} \tilde{I}\tilde{y}(t, \tau, \sigma, \varepsilon) &\equiv \tilde{I} \left(\sum_{k=0}^{\infty} \varepsilon^k y_k(t, \tau, \sigma) \right) \\ &= \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=1, r-s \geq 0}^r R_{r-s}y_s(t, \tau, \sigma) \Big|_{\tau=\varepsilon^{-1}\varphi(t), \sigma=\psi(t, \varepsilon)}. \end{aligned}$$

Although the extension of the operator \tilde{I} is built formally, it is quite possible to use them for calculation of the asymptotic solution $y_{\varepsilon N}(t)$ of the problem (1) of the finite order $N < \infty$.

Due to (11), "extended" problem (7) can be rewritten as follows:

$$\begin{aligned} \tilde{L}_\varepsilon\tilde{y}(t, \tau, \sigma, \varepsilon) &\equiv \varepsilon \frac{\partial \tilde{y}}{\partial t} + D_\lambda \tilde{y} + \lambda_m(t) \frac{\partial \tilde{y}}{\partial \tau_m} + [\lambda_m(t)\sigma + \varepsilon] \frac{\partial \tilde{y}}{\partial \sigma} \\ &\quad - A(t)\tilde{y} + R\tilde{y} = h(t), \quad \tilde{y}(0, 0, 0, \varepsilon) = y^0 \end{aligned} \tag{12}$$

where R integral operator defined above.

Putting the series (10) into the system (12) and equating coefficients of ε with the same powers; we get the problem:

$$Ly_0 \equiv L_0y_0 - R_0y_0 = h(t), \quad y_0(0, 0, 0) = y^0; \tag{13_0}$$

$$Ly_1 \equiv -\frac{\partial y_0}{\partial t} - \frac{\partial y_0}{\partial \sigma} + R_1 y_0, \quad y_1(0, 0, 0) = 0; \tag{13_1}$$

...

$$Ly_k \equiv -\frac{\partial y_{k-1}}{\partial t} - \frac{\partial y_{k-1}}{\partial \sigma} + R_k y_{k-1}, \quad y_k(0, 0, 0) = 0, \quad k \geq 2 \tag{13_k}$$

where $L_0 \equiv \sum_{j=0}^p \lambda_j(t) \frac{\partial}{\partial \tau_j} + \lambda_m(t) \frac{\partial}{\partial \tau_m} + \lambda_m(t) \sigma \frac{\partial}{\partial \sigma} - A(t)$, R_0, R_1, \dots , are integral operators defined by the ratio (11).

3 Solvable iterative problems

Each of the iterative problems of the form

$$(L_0 - R_0)y(t, \tau, \sigma) = h(t, \tau, \sigma), \quad y(0, 0, 0) = y^0 \tag{14}$$

where $h(t, \tau)$ is the corresponding right side.

Theorem 1.2 *Let conditions (i) - (iii) hold, and the right hand side of the system (14) belongs to the space U. Then this system is solvable in U if and only if*

$$\langle h(t, \tau, \sigma), \nu_k(t, \tau, \sigma) \rangle \equiv 0, \quad k = \overline{1, n} \quad \forall t \in [0, T] \tag{15}$$

where $\nu_k(t, \tau, \sigma)$, $k = \overline{1, n}$ are basis of kernel of the conjugate operator

$$L^* \equiv \sum_{j=1}^p \bar{\lambda}_j(t) \frac{\partial}{\partial \tau_j} + \varepsilon \bar{\lambda}_m(t) \frac{\partial}{\partial \tau_m} + \bar{\lambda}_m(t) \sigma \frac{\partial}{\partial \sigma} - A^*(t).$$

Proof Let $h(t, \tau, \sigma) = h_0(t) + \sum_{j=0}^p h_j(t) e^{\tau_j} + h_m(t) e^{\tau_m} + h_{m+1}(t) \sigma$. Define a solution of the system (14) in the form (9). Putting (9) into the system (14) and equating coefficients of e^{τ_j} , e^{τ_m} , σ , and the free terms, we get the following systems:

$$-A(t)y_0(t) = h_0(t), \tag{16}$$

$$[\lambda_s(t)I - A(t)]y_s(t) = h_s(t), \quad s = \overline{1, n}, \tag{17}$$

$$[\lambda_m(t)I - A(t)]y_m(t) + \int_0^t K(t, s)y_m(s)ds = h_m(t), \tag{18}$$

$$[\lambda_m(t)I - A(t)]y_{m+1}(t) + \int_0^t K(t, s)y_{m+1}(s)ds = -K_0(t, 0) + \hat{K}_{m+1}(t, 0) \tag{19}$$

where I is a unique operator.

Consider the system (16). As noted above, this system has a solution of the form (4). System (18) is the Volterra integral equation of the second

type with the kernel $G(t, s) = [\lambda_m(t)I - A(t)]^{-1} K(t, s)$ and the free terms $g_m(t) = [\lambda_m(t)I - A(t)]^{-1} h_m(t)$. It is known, that such equations are uniquely solvable in the class $C^\infty[0, T]$. Likewise, the system of integral equations (19) is a system of Volterra integral equations of the second type with kernel $G(t, s) = [\lambda_m(t)I - A(t)]^{-1} K(t, s)$ and free terms $g_{m+1}(t) = [\lambda_m(t)I - A(t)]^{-1} h_{m+1}(t)$.

To calculate a solution of the system (17) we make transformation

$$y_s(t) = C(t)\xi(t) \equiv \sum_{s=1}^n \xi_s(t)c_s(t) \tag{20}$$

where $\xi(t) = \{\xi_1, \dots, \xi_n\}$ is a new unknown vector, $C(t) = (c_1, \dots, c_n)$ is a matrix of the columns c_k , which are eigenvectors of the operator $A(t)$. Putting (20) into the system (17), we have:

$$[\lambda_s(t)I - A(t)] C(t)\xi(t) = h_s(t).$$

Multiplying from the left this equality by the matrix $C^{-1}(t)$ and taking account that $C^{-1}(t)A(t)C(t) = \Lambda(t) \equiv \text{diag}(\lambda_1, \dots, \lambda_n)$, we get

$$(\lambda_s(t) - \lambda_1(t))\xi_1(t) = (h_s(t), d_1(t)).$$

Since $\lambda_i(t) \equiv 0$, $i = \overline{p+1, n}$ and $C^{-1}(t)h_s(t) = \{(h_s, d_1), \dots, (h_s, d_n)\}$, we write this system componentwise as

$$\begin{aligned} &(\lambda_s(t) - \lambda_1(t)) \xi_1(t) = (h_s(t), d_1(t)), \\ &\dots \\ &(\lambda_s(t) - \lambda_{s-1}(t)) \xi_{s-1}(t) = (h_s(t), d_{s-1}(t)), \\ &0 \cdot \xi_s(t) = (h_s(t), d_{s1}(t)), \\ &(\lambda_s(t) - \lambda_{s+1}(t)) \xi_{s+1}(t) = (h_s(t), d_{s+1}(t)), \\ &\dots \\ &(\lambda_s(t) - \lambda_p(t)) \xi_p(t) = (h_s(t), d_p(t)), \\ &\lambda_s(t) - \lambda_{p+1}(t) = (h_s(t), d_{p+1}(t)), \\ &\dots \\ &\lambda_s(t)\xi_n(t) = (h_s(t), d_n(t)). \end{aligned} \tag{21}$$

The last $(n - p)$ components of the system (21) are uniquely solvable in the class $C^\infty[0, T]$, i.e.

$$y_i(t) = C(t)\xi_i(t) = \sum_{i=p+1}^n \frac{(h_s(t), d_i(t))}{\lambda_s(t)} c_i(t).$$

The first p components of the system (21) are solvable in the class $C^\infty[0, T]$ if and only if $\langle h_k(t), d_j(t) \rangle \equiv 0$, $j = \overline{1, p}$, which coincides with the condition (15). And it has a solution:

$$y_i(t) = \sum_{j=1}^p \left[\alpha_j(t)c_j(t) + \sum_{s=1, s \neq j}^p \frac{(h_s(t), d_j(t))}{\lambda_s(t)} c_j(t) \right]$$

where $\alpha_j(t) \in C^\infty([0, T], C^1)$ are arbitrary functions, $j = \overline{1, p}$. Theorem 1.2 is proved.

Remark When orthogonality conditions (15) hold the system (14) has the following solution

$$\begin{aligned}
 y(t, \tau, \sigma) = & \sum_{j=1}^p \left[\alpha_j(t)c_j(t) + \sum_{s=1, s \neq j}^p \frac{(h_s(t), d_j(t))}{\lambda_s(t) - \lambda_j(t)} c_s(t) \right] e^{\tau_j} \\
 & + \sum_{i=p+1}^n \frac{(h_s(t), d_i(t))}{\lambda_s(t)} c_i(t) + \left[g_m(t) + \int_0^t R(t, s)g_m(s)ds \right] e^{\tau_m} \\
 & + \left[g_{m+1}(t) + \int_0^t R(t, s)g_{m+1}(s)ds \right] \sigma
 \end{aligned} \tag{22}$$

where $\alpha_j(t) \in C^\infty[0, T]$ are scalar functions, $j = \overline{1, p}$, $R(t, s)$ is a resolvent of the kernel $\lambda_m(t)I - A(t)^{-1}K(t, s)$.

Now we study unique solvability of the system (14).

Theorem 1.3 *Let conditions of the theorem 1.2 hold and the right hand side $h(t, \tau, \sigma) \in U$ of the system (14) satisfies orthogonality conditions (15). Then the system (14) when*

$$\left\langle -\frac{\partial y}{\partial t} - \frac{\partial y}{\partial \sigma} + R_1 y + Q(t, \tau, \sigma), \nu_k(t, \tau, \sigma) \right\rangle \equiv 0, \quad \forall t \in [0, T], \quad k = \overline{1, n} \tag{23}$$

where $Q(t, \tau, \sigma)$ is a given operator and R_1 is integral operator defined above, has a unique solution in the space U .

Proof. Since conditions (15) hold, the system (14) has a solution in the space U which can be represented in the form (see. (22)):

$$y(t, \tau, \sigma) = \sum_{j=1}^p \alpha_j(t)c_j(t)e^{\tau_j} + \hat{y}(t, \tau, \sigma) \tag{24}$$

where

$$\begin{aligned}
 \hat{y}(t, \tau, \sigma) = & \left[g_m(t) + \int_0^t R(t, s)g_m(s)ds \right] e^{\tau_m} \\
 & + \left[g_{m+1}(t) + \int_0^t R(t, s)g_{m+1}(s)ds \right] \sigma + \sum_{i=p+1}^n \frac{(h_s(t), d_i(t))}{\lambda_s(t)} c_i(t) \\
 & + \sum_{j=1}^p \sum_{s=1, s \neq j}^p \frac{(h_s(t), d_j(t))}{\lambda_s(t) - \lambda_j(t)} c_j(t)e^{\tau_j} \equiv \sum_{j=1}^p \hat{y}_j(t)e^{\tau_j} + \hat{y}_m(t)e^{\tau_m} + \hat{y}_{m+1}(t)\sigma + \hat{y}_0(t).
 \end{aligned}$$

Subjecting the solution (24) to the initial conditions $y(0, 0, 0) = y^0$, we have

$$\sum_{j=1}^p \alpha_j(0)c_j(0) = y^0 - \hat{y}(0, 0, 0).$$

Multiplying by a scalar this equality to $d_k(t)$, $k = \overline{1, n}$, we find value of $\alpha_j(0)$:

$$\alpha_j(0) = (y^0, d_k(0)) - (\hat{y}(0, 0, 0), d_k(0)), \quad j = \overline{1, p}, \quad k = \overline{1, n}. \quad (25)$$

Since

$$R_1 y(t, \tau, \sigma) = \frac{K_0^1(t, t)}{\mu(t)} - \frac{K_0^1(t, 0)}{\mu(0)} e^{\tau_m} + \frac{K_j(t, t)}{\lambda_j(t) - \lambda_m(t)} e^{\tau_j} - \frac{K_j(t, 0)}{\lambda_j(0) - \lambda_m(0)} e^{\tau_m} + \frac{\hat{K}_{m+1}^1(t, t)}{\mu(t)} - \frac{\hat{K}_{m+1}^1(t, 0)}{\mu(0)} e^{\tau_m}$$

we write orthogonality conditions (23) as follows:

$$\left\langle - \sum_{j=1}^p (\alpha_j(t) c_j(t)) e^{\tau_j} - \frac{\partial \hat{y}(t, \tau, \sigma)}{\partial t} - \frac{\partial \hat{y}(t, \tau, \sigma)}{\partial \sigma} - \sum_{j=1}^p \frac{K(t, t) \alpha_j(t)}{\lambda_j(t) - \lambda_m(t)} c_j(t) e^{\tau_j} + R_1 \hat{y}(t, \tau, \sigma) + Q(t, \tau, \sigma), d_k(t) e^{\tau_k} \right\rangle = 0, \quad k = \overline{1, n}.$$

Multiplying by the scalar product, we obtain ordinary differential equations for the unknown functions $\alpha_j(t)$:

$$-\dot{\alpha}_j(t) - \left[(\dot{c}_j(t), d_k(t)) - \frac{K(t, t) c_j(t)}{\lambda_j(t) - \lambda_m(t)} \right] \alpha_j(t) = 0, \quad j = \overline{1, p}, \quad k = \overline{1, n}.$$

Adding the initial conditions (25) to these equations, we find the function $\alpha_j(t)$ uniquely:

$$\alpha_j(t) = \left[(y^0, d_k(0)) - (\hat{y}(0, 0, 0), d_k(0)) \right] e^{-\int_0^t G(s) ds}$$

where $G(s) = \left[(\dot{c}_j(t), d_k(t)) - \frac{K(t, t) c_j(t)}{\lambda_j(t) - \lambda_m(t)} \right]$. Thus the solution (22) of the system (14) is uniquely defined in the space U . Theorem 1.3 is proved.

4 Asymptotic characteristics of the formal solutions

The N -th partial sum of the series (12) we denote by $S_N(t, \tau, \sigma, \varepsilon) = \sum_{k=0}^N \varepsilon^k y_k(t, \tau, \sigma)$, and by $y_{\varepsilon N}(t) = S_N(t, \frac{\varphi(t)}{\varepsilon}, \psi(t, \varepsilon), \varepsilon)$ restriction of this series when $\tau = \frac{\varphi(t)}{\varepsilon}$, $\sigma = \psi(t, \varepsilon)$.

Firstly we prove the following proposition.

Lemma Let conditions (i) - (iii) hold. Then the function $y_{\varepsilon N}(t)$ satisfies the problem (1) up to terms containing ε^{N+1} , i.e.

$$\varepsilon \frac{dy_{\varepsilon N}(t)}{dt} = A(t) y_{\varepsilon N} + \int_0^t e^{\frac{1}{\varepsilon} \int_s^t \mu(x) dx} K(t, s) y_{\varepsilon N}(s, \varepsilon) ds$$

$$+h(t) + \varepsilon^{N+1}F_n(t, \varepsilon), \quad y_{\varepsilon N}(0) = 0 \quad (26)$$

where $\|F_N(t, \varepsilon)\|_{C[0, T]} \leq \bar{F}_N$, $\bar{F}_N > 0$ – is a constant, which does not depend on $\forall(t, \varepsilon) \in [0, T] \times (0, \varepsilon_0]$; $\varepsilon_0 > 0$ – infinitesimal.

Proof Put solutions $y_0(t, \tau, \sigma), \dots, y_N(t, \tau, \sigma)$ into iterative (13₀), \dots , (13_N), multiply this identity by $1, \varepsilon, \dots, \varepsilon^N$, relatively, and add the results. Then we get:

$$\begin{aligned} y_N(t, \tau, \sigma, \varepsilon) - R_0 S_N(t, \tau, \sigma, \varepsilon) &\equiv h(t) - \varepsilon \frac{\partial S_N(t, \tau, \sigma, \varepsilon)}{\partial t} \\ &+ \varepsilon^{N+1} \frac{\partial y_{\varepsilon N}(t, \tau, \sigma)}{\partial t} - \varepsilon \frac{\partial S_N(t, \tau, \sigma, \varepsilon)}{\partial \sigma} + \varepsilon R_1 y_0(t, \tau, \sigma) \\ &+ \varepsilon^2 (R_1 y_1 + R_2 y_0) + \dots + \varepsilon^N \sum_{s=0, N-s>0}^{N-1} R_{N-s}. \end{aligned}$$

or

$$\begin{aligned} y_N(t, \tau, \sigma, \varepsilon) + \varepsilon \frac{\partial S_N(t, \tau, \sigma, \varepsilon)}{\partial t} &\equiv h(t) + \varepsilon^{N+1} \frac{\partial y(t, \tau, \sigma)}{\partial t} - \varepsilon \frac{\partial S_N(t, \tau, \sigma)}{\partial \sigma} \\ &+ R_0 y_0 + \varepsilon (R_0 y_1 + R_1 y_0) + \varepsilon^2 (R_0 y_2 + R_1 y_1 + R_2 y_0) \\ &+ \dots + \varepsilon^N (R_0 y_N + R_1 y_{N-1} + \dots + R_N y_0). \end{aligned}$$

Subtracting from both sides of this identity

$$I S_N(t, \tau, \sigma, \varepsilon) \equiv I y_{\varepsilon N}(t)$$

we make the restriction when $\tau = \varphi(t, \varepsilon)$ and $\sigma = \psi(t, \varepsilon)$, then we have

$$\begin{aligned} -A(t)y_{\varepsilon N} - I y_{\varepsilon N}(t) &\equiv h(t) + \varepsilon^{N+1} \frac{\partial y_N(t, \varphi(t)/\varepsilon, \psi(t, \varepsilon))}{\partial t} - \\ &- \left[I S_N(t, \tau, \sigma, \varepsilon) - \sum_{r=0}^N \varepsilon^r \sum_{s=0, r-s \geq 0}^r R_{r-s}(t, \tau, \sigma) \Big|_{\tau=\varphi(t, \varepsilon), \sigma=\psi(t, \varepsilon)} \right]. \end{aligned}$$

By construction of the operator R_m expression in the square brackets can be represented as $\varepsilon^{N+1}l_N(t, \varepsilon)$ and $\|l_N(t, \varepsilon)\|_{C[0, T]} \leq \bar{l}_N = const$. Consequently,

$$\varepsilon \frac{dy_{\varepsilon N}(t)}{dt} - A(t)y_{\varepsilon N} - I y_{\varepsilon N}(t) \equiv h(t) + \varepsilon^{N+1}F_N(t, \varepsilon)$$

where $F_N(t, \varepsilon) \equiv \frac{\partial y_N(t, \varphi(t, \varepsilon), \psi(t, \varepsilon))}{\partial t} - l_N(t, \varepsilon)$. Due to the conditions (i) - (iii), the vector valued function $F_N(t, \varepsilon)$ is uniformly bounded, i.e.

$$\|F_N(t, \varepsilon)\|_{C[0, T]} \leq \bar{F}_N = const \quad \forall(t, \varepsilon) \in [0, T] \times (0, \varepsilon_0]$$

where $\varepsilon_0 > 0$ is infinitesimal.

Correctness of the equality $y_{\varepsilon N}(0) = 0$ is obviously. Then all $y_j(t, \tau, \sigma)$ satisfy the condition $y_j(0, 0, 0) = 0$. Lemma is proved.

Using this lemma, it is easy to prove the following statement.

Theorem 1.4 *Let conditions (i) - (iii) hold. Then the problem (1) is uniquely solvable in the class $C^1([0, T], C^{N-1})$ and for its solution $y(t, \varepsilon)$ the following estimation holds:*

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C[0, T]} \leq C_N \varepsilon^{n+1}$$

where constants $C_N > 0$ do not depend on ε when $(0, \varepsilon_0]$ ($\varepsilon_0 > 0$ infinitesimal).

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