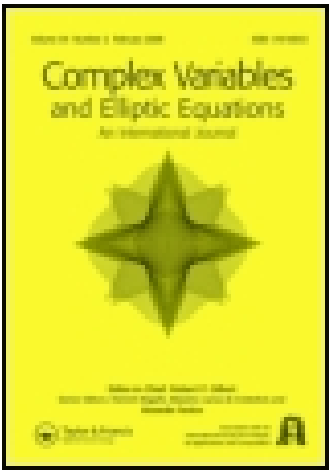

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Representation of Green's function of the Neumann problem for a multi-dimensional ball

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Representation of the Green's function of the classical Neumann problem for the Poisson equation in the unit ball of arbitrary dimension is given. In constructing this function, we use the representation of the fundamental solution of the Laplace equation in the form of a series. It is shown that the Green's function can be represented in terms of elementary functions and its explicit form can be written out. An explicit form of the Neumann kernel at $n = 4$ and $n = 5$.

Keywords: Poisson equation; Laplace operator; Neumann problem; fundamental solutions; Green's function

AMS Subject Classifications: 35J05; 35J08; 35J25; 65N80

1. Introduction

The Dirichlet problem for the Poisson equation

$$-\Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u(x) = f(x), \ x \in D; \quad u = \varphi, \ x \in \partial D, \tag{1}$$

in the domain $D \subset R^n$, $n \geq 2$ with the regular boundary ∂D is the classic and well-investigated problem. The solution of the problem (1) exists, is unique and is represented by the Green's function $G_D(x, y)$ in the form (see [1, p.35]):

$$u(x) = \int_D G_D(x, y) f(y) \, \mathrm{d}y - \int_{\partial D} \frac{\partial G_D}{\partial n_y}(x, y) \varphi(y) \, \mathrm{d}S_y. \tag{2}$$

Here and further, $\frac{\partial}{\partial n}$ is a derivative in the direction of an outer normal to ∂D.

In case, $D = \{x \in R^n : |x| < 1\}$ is the unit ball, the Green's function of the Dirichlet problem can be constructed by reflection method and has the form:

$$G_D(x, y) = \frac{1}{\omega_n} \left[\varepsilon(x - y) - \varepsilon\left(x|y| - \frac{y}{|y|} \right) \right], \tag{3}$$

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where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the unit sphere in R^n, and $\varepsilon(x-y)$ is the fundamental solution of the Laplace equation:

$$\varepsilon(x-y) = \begin{cases} -\ln|x-y|, & n=2; \\ \frac{1}{n-2}|x-y|^{2-n}, & n \geq 3. \end{cases} \tag{4}$$

Along with the Dirichlet problem, the Neumann problem for the Poisson equation is classic and well investigated:

$$-\Delta u(x) \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u(x) = f(x), \ x \in D; \quad \frac{\partial u}{\partial n} = \psi, \ x \in \partial D. \tag{5}$$

It is well known that the solution of the Neumann problem (5) is not unique up to a constant summand. The fulfilment of the following condition

$$\int_D f(y)\,\mathrm{d}y + \int_{\partial D} \psi(y)\,\mathrm{d}S_y = 0. \tag{6}$$

is necessary and sufficient for existence of the problem solution.

If the solution of the problem (5) exists then this solution can be represented in the integral form by means of the Green's function of the Neumann problem $G_N(x,y)$ by formula similar to the representation (2) (see [2, p.286]):

$$u(x) = \int_D G_N(x,y) f(y)\,\mathrm{d}y + \int_{\partial D} G_N(x,y)\psi(y)\,\mathrm{d}S_y + \mathrm{Const}. \tag{7}$$

Although the definition of the Green's function of the Neumann problem exists in the mathematical literature, it is recognized that finding the Green's function requires a rather complicated construction [2, p.348–350], [3, p.7–8], [4, p.244–247], [5, p.296–300], [6].

The Green's function of the Neumann problem (5) is understood as the function (see [2, p.286]) having the representation

$$G_N(x,y) = \frac{1}{\omega_n}\left[\varepsilon(x-y) + g(x,y)\right], \tag{8}$$

where $g(x,y)$ is a harmonic function in the domain D. At the same time the boundary condition

$$\frac{\partial G_N}{\partial n_y}(x,y) = -\frac{1}{\omega_n}, \quad y \in \partial D. \tag{9}$$

must be held. If such Green's function $G_N(x,y)$ exists then from (6) and (9) easily follows that function (7) satisfies all the conditions of problem (5). In case of an arbitrary domain D, the existence of the Green's function of the Neumann problem is reduced to a second-order Fredholm integral equation relatively to unknown function $g(x,y)$. Green's function of the Neumann problem (Green's function of the second kind) is also called as Neumann function.

Various construction methods of the Green's function of the Dirichlet problem (1) exist. The Green's function was constructed in the explicit form for many types of the domain D. But for the Neumann problem (5) in the multidimensional spaces the construction of the Green's function is an open problem. Here we have only samples for the simplest domains – halfspaces,[7] quarters of space,[8] a semicircle [9] etc. For such domains, the Neumann problem is an outer boundary value problem and therefore will be correct without fulfilment of solvability conditions of the form (6). For the unit ball from R^n, the Green's function of the Neumann problem has been constructed in the explicit form only for cases $n = 2$ and $n = 3$:

$$G_N(x, y) = \frac{1}{2\pi}\left[\ln\frac{1}{|x-y|} + \ln\frac{1}{\left|x\,|y| - \frac{y}{|y|}\right|}\right], \quad n = 2;$$

$$G_N(x, y) = \frac{1}{4\pi}\left[\frac{1}{|x-y|} + \frac{1}{\left|x\,|y| - \frac{y}{|y|}\right|} - \ln\left|1 - (x, y) + \left|x\,|y| - \frac{y}{|y|}\right|\right|\right], \quad n = 3; \tag{10}$$

where $(x, y) = x_1 y_1 + \cdots + x_n y_n$ is a scalar product of the vectors x and y in R^n.

Note that the interest to the construction of the Green's function of classical problems in the explicit form lately renewed. The Green's functions of the Dirichlet, Neumann and Robin biharmonic problems in a two-dimensional disc were constructed by means of the Green's harmonic functions of the Dirichlet problem in [10]. The similar results in the class of nonuniform biharmonic and triharmonic functions in a sector were obtained in [11,12]. The Green's function of the Dirichlet problem for a polyharmonic equation in the multidimensional ball was constructed in the explicit form in [13–17]. We also note that works [18–20] are devoted to the construction of the Green's function of the Robin problem in the explicit form in a circle.

In the present paper, we will give the representation of the Green's function of the Neumann problem for unit ball of arbitrary dimension in the explicit form (in the terms of elementary functions). It is shown that the function can be represented in terms of elementary functions and its explicit form can be written out in particular cases.

2. Auxiliary results

LEMMA 2.1 *For fundamental solution* (4) *of the Laplace operator, we have the representations*

$$\varepsilon(x-y) = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|x|^k}{|y|^{k+n-2}}\sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right), \quad \text{if } |x| < |y|, \tag{11}$$

$$\varepsilon(x-y) = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|y|^k}{|x|^{k+n-2}}\sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right), \quad \text{if } |x| > |y|, \tag{12}$$

where $H_k^{(i)}(\cdot)$ *is a full system of homogeneous harmonic polynomials of* k *degree having the property of orthonormality:*

$$\frac{1}{\omega_n}\int_{\partial\Omega} H_k^{(i)}(x)\, H_m^{(j)}(x)\, \mathrm{d}S_x = \delta_{ij}\delta_{km}, \tag{13}$$

and h_k *is the quantity of these polynomials.*

The number h_k *is obtained by the formula (see* [21, p.237, §11.2. formula (2)]*)* $h_k = [(2k+n-2)(k+n-3)!] / [k!(n-2)!]$.

Proof Consider the first case. The second case is proved similarly. Since $|x| < |y|$, then

$$|x-y|^{2-n} = \left| \frac{x}{|y|}|y| - \frac{y}{|y|}|y| \right|^{2-n} = |y|^{2-n} \left| \frac{x}{|y|} - \frac{y}{|y|} \right|^{2-n} = |y|^{2-n} |\xi - \eta|^{2-n},$$

where $\xi = \frac{x}{|y|}$, $\eta = \frac{y}{|y|}$, $|\xi| < 1$, $|\eta| = 1$.

Consider Gegenbauer polynomials $C_n^{\nu}(t)$, obtained as coefficients of expansion of the function $(1-2\eta t+\eta^2)^{-\nu}$ in a series (see e.g. [21, p.235, formula (16)]]):

$$(1-2\eta t+\eta^2)^{-\nu} = \sum_{k=0}^{\infty} C_k^{\nu}(t)\eta^k.$$

Setting here $\nu = \frac{n}{2}-1$ i.e. $-\nu = 1-\frac{n}{2}$, we have

$$\left[1-2\left(\frac{x}{|y|},\frac{y}{|y|}\right)+\frac{|y|^2}{|y|^2}\right]^{1-\frac{n}{2}} = \left[1-2\frac{|x|}{|y|}\left(\frac{x}{|x|},\frac{y}{|y|}\right)+\frac{|x|^2}{|y|^2}\right]^{1-\frac{n}{2}}$$
$$= \sum_{k=0}^{\infty} C_k^{\frac{n}{2}-1}\left[\left(\frac{x}{|x|},\frac{y}{|y|}\right)\right]\frac{|y|^k}{|x|^k}$$

when $t = \left(\frac{x}{|x|},\frac{y}{|y|}\right)$.

Consequently, since

$$|\xi-\eta| = \sqrt{|\xi|^2-2(\xi,\eta)+|\eta|^2} = \sqrt{1-2\left(\frac{x}{|x|},\frac{y}{|y|}\right)+\frac{|x|^2}{|y|^2}},$$

then

$$|\xi-\eta|^{2-n} = \sum_{k=0}^{\infty} C_k^{\frac{n}{2}-1}\left(\left(\frac{x}{|x|},\frac{y}{|y|}\right)\right)\frac{|x|^k}{|y|^k}.$$

For the Gegenbauer polynomials the following relations (see [21, p.236, formula (28)]):

$$C_k^{\frac{n}{2}-1}(1) = \frac{(k+n-3)!}{k!(n-3)!}$$

hold. Moreover, if $H_k^{(i)}(x), i = 1,\ldots,h_k$ is a full system of homogeneous harmonic polynomials of k degree satisfying the normalization condition (13), then the number h_k is obtained by the formula (see [20, p.237, §11.2. formula (2)]) $h_k = (2k+n-2)\frac{(k+n-3)!}{k!(n-2)!}$. Then between the numbers h_k and $C_k^{\frac{n}{2}-1}(1)$ the following relation $h_k = \frac{2k+n-2}{n-2}C_k^{\frac{n}{2}-1}(1)$ holds.

Further, if $S_k^{(i)}(\xi)$ is an arbitrary orthonormal system of spherical harmonics of degree k, then (see e.g. [20, p.243, formula (2)])

$$\frac{C_k^{\frac{n}{2}-1}((\xi,\eta))}{C_k^{\frac{n}{2}-1}(1)} = \frac{\omega_n}{h_k}\sum_{i=1}^{h_k} S_k^{(i)}(\xi)S_k^{(i)}(\eta),$$

when $|\xi| = 1, \ |\eta| = 1$. Then the function $|x-y|^{2-n}$ is represented in the form

$$|x-y|^{2-n} = |y|^{2-n} \sum_{k=0}^{\infty} C_k^{\frac{n}{2}-1} \left[\left(\frac{x}{|x|}, \frac{y}{|y|} \right) \right] \frac{C_k^{\frac{n}{2}-1}(1)}{C_k^{\frac{n}{2}-1}(1)} \frac{|x|^k}{|y|^k}$$

$$= \sum_{k=0}^{\infty} C_k^{\frac{n}{2}-1}(1) \left(\frac{\omega_n}{h_k} \sum_{i=1}^{h_k} S_k^{(i)}(\xi) S_k^{(i)}(\eta) \right) \frac{|x|^k}{|y|^{k+n-2}}$$

$$= \omega_n \sum_{k=0}^{\infty} \left(\frac{C_k^{\frac{n}{2}-1}(1)}{h_k} \sum_{i=1}^{h_k} S_k^{(i)}(\xi) S_k^{(i)}(\eta) \right) \frac{|x|^k}{|y|^{k+n-2}}$$

$$= \omega_n \sum_{k=0}^{\infty} \left(\frac{n-2}{2k+n-2} \sum_{i=1}^{h_k} S_k^{(i)}(\xi) S_k^{(i)}(\eta) \right) \frac{|x|^k}{|y|^{k+n-2}}.$$

Let us choose the spherical harmonics $S_k^{(i)}(\xi)$ from the conditions:

$$H_k^{(i)}(\xi) = \sqrt{\omega_n} S_k^{(i)}(\xi), \ H_k^{(i)}(\eta) = \sqrt{\omega_n} S_k^{(i)}(\eta).$$

Then

$$|x-y|^{2-n} = \omega_n \sum_{k=0}^{\infty} \left(\frac{n-2}{2k+n-2} \sum_{i=1}^{h_k} \frac{H_k^{(i)}(\xi)}{\sqrt{\omega_n}} \frac{H_k^{(i)}(\eta)}{\sqrt{\omega_n}} \right) \frac{|x|^k}{|y|^{k+n-2}}$$

$$= \sum_{k=0}^{\infty} \left(\frac{n-2}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}(\xi) H_k^{(i)}(\eta) \right) \frac{|x|^k}{|y|^{k+n-2}}.$$

Therefore, for the function $\varepsilon(x-y)$ we get the representation (12). Lemma 2.1 is proved completely. □

Consider the function ($n \geq 3$):

$$\varepsilon_1(x, y) = \int_0^1 \left[(n-2)\, \varepsilon \left(sx\,|y| - \frac{y}{|y|} \right) - 1 \right] \frac{\mathrm{d}s}{s} \equiv \int_0^1 \left[\left| sx\,|y| - \frac{y}{|y|} \right|^{2-n} - 1 \right] \frac{\mathrm{d}s}{s}. \tag{14}$$

LEMMA 2.2 *Let $n \geq 3$, then for the function $\varepsilon_1(x, y)$ given by Equation (14), we have the representations in the elementary functions:*

(i) $\varepsilon_1(x, y) = \ln \frac{2}{\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|}$ *when* $n = 3$;

(ii) $\varepsilon_1(x, y) = \frac{(x,y)}{\sqrt{|x|^2|y|^2-(x,y)^2}} \arctan \frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{1-(x,y)} - \ln \left| x\,|y| - \frac{y}{|y|} \right|$ *when* $n = 4$;

(iii) $\varepsilon_1(x, y) = \ln \frac{2}{1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|} + \sum_{k=1}^{m-1} \frac{1}{(2k-1)} \left(\left| x\,|y| - \frac{y}{|y|} \right|^{1-2k} - 1 \right)$

$$+\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1}\frac{2^i(k+i-1)!(2k-3)!!}{(k-1)!(2k+2i-1)!!}\frac{(x,y)|x|^{2i}|y|^{2i}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i+1}}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right),$$

when $n\geq 5, n=2m+1, m\geq 2$;

(iv) $\varepsilon_1(x,y) = -\ln\left|x|y|-\frac{y}{|y|}\right| + \sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x|y|-\frac{y}{|y|}\right|^{-2k}-1\right) + (x,y)$ $\arctan\frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{(1-(x,y))}\sum_{k=0}^{m-1}\frac{(2k-1)!!}{2^k k!}\frac{|x|^{2k}|y|^{2k}}{\left(|x|^2|y|^2-(x,y)^2\right)^{k+\frac{1}{2}}}$

$$+\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1}\frac{(2k+2i-1)!!(k+1)!}{2^{i+1}(2k-1)!!(k+i)!}\frac{(x,y)|x|^{2i}|y|^{2i}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i+1}}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k}}-(x,y)\right),$$

when $n\geq 6, n=2m+2, m\geq 2$. *Here, in the notations it is accepted that* $0!=1, (-1)!!=1$.

Proof Taking into account that

$$\left|sx|y|-\frac{y}{|y|}\right|=\sqrt{1-2(x,y)s+|x|^2|y|^2s^2},$$

the proof of the lemma is based on the calculation of integrals of the form

$$\int_0^1 s^{-1}R(s)^{\frac{2-n}{2}}\,\mathrm{d}s,$$

where $R(s)=1-2(x,y)s+|x|^2|y|^2s^2$, by using formulas from [22].

Let us consider each case separately.

The first case: Let us show that (i) is true when $n=3$. Here after we use some formulas from [22]. Denote

$$R(s)=a+bs+cs^2,\ \Delta=4ac-b^2. \tag{15}$$

where

$$a=1, b=-2(x,y), c=|x|^2|y|^2, \Delta=4\left(|x|^2|y|^2-(x,y)^2\right). \tag{16}$$

Then (see [22, p.97, example 2.266, formula (137)])

$$\int\frac{1}{\sqrt{R(s)}}\frac{\mathrm{d}s}{s}=-\frac{1}{\sqrt{a}}\ln\frac{2a+bs+2\sqrt{aR(s)}}{s},\quad a>0.$$

Substituting here

$$R(s)=a+bs+cs^2=1-2(x,y)s+|x|^2|y|^2s^2=\left|sx|y|-\frac{y}{|y|}\right|^2,$$

then for the function $\varepsilon_1(x,y)$ in case $n=3$ we have

$$\int_0^1\left[\frac{1}{\sqrt{1-2s(x,y)+s^2|x|^2|y|^2}}-1\right]\frac{\mathrm{d}s}{s}$$
$$=-\left(\ln\frac{2-2(x,y)s+2\left|sx|y|-y/|y|\right|}{s}+\ln s\right)\Bigg|_{s=0}^{s=1}$$
$$=-\left(\ln\left|2-2(x,y)s+2\left|sx|y|-\frac{y}{|y|}\right|\right|\right)\Bigg|_{s=0}^{s=1}$$

$$= \ln \frac{2}{1-(x,y)+\left|x\,|y| - \frac{y}{|y|}\right|}.$$

That is, when $n = 3$ we have the representation (i) of the function $\varepsilon_1(x, y)$.

The second case: Let us show that (ii) is true for $n = 4$. We use the equation (see [22, p.80, example 2.177, formula 1]):

$$\int \frac{1}{R}\frac{\mathrm{d}s}{s} = \frac{1}{2a}\ln\frac{s^2}{R} - \frac{b}{2a}\int\frac{\mathrm{d}s}{R}.$$

Here we use (see [22, p.79, example 2.172, case $\Delta > 0$]):

$$\int\frac{\mathrm{d}s}{R} = \frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}},$$

we get

$$\int\frac{1}{R}\frac{\mathrm{d}s}{s} = \frac{1}{2a}\ln\frac{s^2}{R} - \frac{b}{a\sqrt{\Delta}}\arctan\frac{b+2sc}{\sqrt{\Delta}}.$$

In total, for the function $\varepsilon_1(x, y)$ when $n = 4$, taking into account $a = 1$, $b = -2(x, y)$, $c = |x|^2\,|y|^2$, $\Delta = 4\left(|x|^2\,|y|^2 - (x, y)^2\right)$, we will have:

$$\varepsilon_1(x,y) = \int_0^1\left[\frac{1}{1-2(x,y)s+|x|^2\,|y|^2\,s^2} - 1\right]\frac{\mathrm{d}s}{s} = \int_0^1\left[\frac{1}{sR} - \frac{1}{s}\right]\mathrm{d}s$$

$$= \left(\frac{1}{2}\ln\frac{s^2}{\left|sx\,|y| - \frac{y}{|y|}\right|^2} + \frac{(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\arctan\frac{s\,|x|^2\,|y|^2-(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}} - \ln s\right)\Bigg|_{s=0}^{s=1}$$

$$= \left(\ln\frac{1}{\left|sx\,|y| - \frac{y}{|y|}\right|} + \frac{(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\arctan\frac{s\,|x|^2\,|y|^2-(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\right)\Bigg|_{s=0}^{s=1} = \ln\frac{1}{\left|x\,|y| - \frac{y}{|y|}\right|}$$

$$+ \frac{(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\left(\arctan\frac{|x|^2\,|y|^2-(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}} - \arctan\frac{-(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\right).$$

here we use: $\arctan A - \arctan B = \arctan\frac{A-B}{1+AB}$, then

$$\varepsilon_1(x,y) = -\ln\left|x\,|y| - \frac{y}{|y|}\right| + \frac{(x,y)}{\sqrt{|x|^2\,|y|^2-(x,y)^2}}\left(\arctan\frac{\sqrt{|x|^2\,|y|^2-(x,y)^2}}{1-(x,y)}\right).$$

So, we get the representation (ii) of the function $\varepsilon_1(x, y)$ for $n = 4$.

The third case: Further, let us consider separately even and odd values of n at $n \geq 5$. We will show that if $n \geq 5$, $n = 2m + 1$, $m \geq 2$, then there is the representation (iii) for the function $\varepsilon_1(x, y)$.

For this we calculate the indefinite integral:

$$\int_0^1\left[\left|sx\,|y| - \frac{y}{|y|}\right|^{1-2m}\right]\frac{\mathrm{d}s}{s} = \int_0^1\frac{\mathrm{d}s}{s\sqrt{R(s)}^{2m-1}}.$$

Using the equation (see [22, p.98, example 2.268, formula (199)]):

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \frac{1}{(2m-3)\,a\sqrt{R^{2m-3}}} - \frac{b}{2a}\int \frac{\mathrm{d}s}{\sqrt{R^{2m-1}}} + \frac{1}{a}\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-3}}}, \quad m \geq 2,$$

we get

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \frac{1}{a^{m-1}}\int \frac{\mathrm{d}s}{s\sqrt{R}} + \sum_{k=0}^{m-2}\left(\frac{1}{(2m-2k-3)\,a^{k+1}\sqrt{R^{2m-2k-3}}} - \frac{b}{2a^{k+1}}\int \frac{\mathrm{d}s}{\sqrt{R^{2m-2k-1}}}\right).$$

Here, we change the index of summation: $j = m-k-1$. Then

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \sum_{j=1}^{m-1}\left(\frac{1}{(2j-1)\,a^{m-j}\sqrt{R^{2j-1}}} - \frac{b}{2a^{m-j}}\int \frac{\mathrm{d}s}{\sqrt{R^{2j+1}}}\right) + \frac{1}{a^{m-1}}\int \frac{\mathrm{d}s}{s\sqrt{R}}.$$

For convenience we return to old notations of summation index: $k := j$:

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \sum_{k=1}^{m-1}\left(\frac{1}{(2k-1)\,a^{m-k}\sqrt{R^{2k-1}}} - \frac{b}{2a^{m-k}}\int \frac{\mathrm{d}s}{\sqrt{R^{2k+1}}}\right) + \frac{1}{a^{m-1}}\int \frac{\mathrm{d}s}{s\sqrt{R}}.$$

For the second summand under the summation sign, we use ([22, p.96, example 2.263, formula 1(191)]):

$$\int \frac{\mathrm{d}s}{\sqrt{R^{2k+1}}} = \frac{2\,(2cs+b)}{(2k-1)\,\Delta\sqrt{R^{2k-1}}}\left\{1 + \sum_{i=1}^{k-1}\frac{8^i\,(k-1)\ldots(k-i)}{(2k-3)\ldots(2k-2i-1)}\frac{c^i}{\Delta^i}R^i\right\}, \quad k \geq 1.$$

And the integral in the last summand has been already calculated under the proof of the first case:

$$\int \frac{1}{\sqrt{R\,(s)}}\frac{\mathrm{d}s}{s} = -\frac{1}{\sqrt{a}}\ln\frac{2a+bs+2\sqrt{aR\,(s)}}{s}, \quad a > 0.$$

So, combining all the calculated integrals, we get:

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \sum_{k=1}^{m-1}\frac{1}{(2k-1)\,a^{m-k}\sqrt{R^{2k-1}}} - \frac{1}{\sqrt{a^{2m-1}}}\ln\frac{2a+bs+2\sqrt{aR}}{s} - \sum_{k=1}^{m-1}\frac{b}{2a^{m-k}}\frac{2\,(2cs+b)}{(2k-1)\,\Delta\sqrt{R^{2k-1}}}\left\{1 + \sum_{i=1}^{k-1}\frac{8^i\,(k-1)\ldots(k-i)}{(2k-3)\ldots(2k-2i-1)}\frac{c^i}{\Delta^i}R^i\right\}$$

we divide the last summand in two parts. At the same time we pay attention that in the inner sum there is a multiplier $(k-1)$ vanishing at $k=1$, then

$$\begin{aligned}\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} &= \sum_{k=1}^{m-1} \frac{1}{(2k-1)\,a^{m-k}\sqrt{R^{2k-1}}} - \frac{1}{\sqrt{a^{2m-1}}} \ln \frac{2a+bs+2\sqrt{aR}}{s} \\ &\quad - \sum_{k=1}^{m-1} \frac{b}{2a^{m-k}} \frac{2\,(2cs+b)}{(2k-1)\,\Delta\sqrt{R^{2k-1}}} - \sum_{k=2}^{m-1} \frac{b}{a^{m-k}\,(2k-1)} \\ &\quad \times \sum_{i=1}^{k-1} \frac{8^i\,(k-1)\ldots(k-i)}{(2k-3)\ldots(2k-2i-1)} \frac{c^i}{\Delta^{i+1}} \frac{(2cs+b)}{\sqrt{R^{2k-2i-1}}}\end{aligned}$$

we change the order of summation in the last summand. We write it separately:

$$\begin{aligned}&-\sum_{k=2}^{m-1} \frac{b}{a^{m-k}\,(2k-1)} \sum_{i=1}^{k-1} \frac{8^i\,(k-1)\ldots(k-i)}{(2k-3)\ldots(2k-2i-1)} \frac{c^i}{\Delta^{i+1}} \frac{(2cs+b)}{\sqrt{R^{2k-2i-1}}} \\ &= -\sum_{k=2}^{m-1}\sum_{i=1}^{k-1} \frac{b}{a^{m-k}\,(2k-1)} \frac{8^i\,(k-1)\ldots(k-i)}{(2k-3)\ldots(2k-2i-1)} \frac{c^i}{\Delta^{i+1}} \frac{(2cs+b)}{\sqrt{R^{2k-2i-1}}} \\ &= \left| \begin{array}{l} k-i=j, i=k-j \\ i=1 \to j=k-1 \\ i=k-1 \to j=1 \end{array} \right| \\ &= -\sum_{k=2}^{m-1}\sum_{j=1}^{k-1} \frac{8^{k-j}b}{a^{m-k}} \frac{(k-1)\ldots(j+1)\,j}{(2k-1)\,(2k-3)\ldots(2j-1)} \frac{c^{k-j}}{\Delta^{k-j+1}} \frac{(2cs+b)}{\sqrt{R^{2j-1}}} \\ &= -\sum_{j=1}^{m-2}\sum_{k=j+1}^{m-1} \frac{8^{k-j}b}{a^{m-k}} \frac{(k-1)\ldots(j+1)\,j}{(2k-1)\,(2k-3)\ldots(2j-1)} \frac{c^{k-j}}{\Delta^{k-j+1}} \frac{(2cs+b)}{\sqrt{R^{2j-1}}} \\ &= -\sum_{j=1}^{m-2}\left(\sum_{k=j+1}^{m-1} \frac{8^{k-j}b}{a^{m-k}} \frac{(k-1)\ldots(j+1)\,j}{(2k-1)\,(2k-3)\ldots(2j-1)} \frac{c^{k-j}}{\Delta^{k-j+1}}\right) \frac{(2cs+b)}{\sqrt{R^{2j-1}}}\end{aligned}$$

for uniformity with the other summands we redenote the indexes of summation: we will write i instead of k in the inner sum, and we will write k instead of j in the outer sum. Then

$$\begin{aligned}&= -\sum_{j=1}^{m-2}\left(\sum_{i=j+1}^{m-1} \frac{8^{i-j}b}{a^{m-i}} \frac{(i-1)\ldots(j+1)\,j}{(2i-1)\,(2i-3)\ldots(2j-1)} \frac{c^{i-j}}{\Delta^{i-j+1}}\right) \frac{(2cs+b)}{\sqrt{R^{2j-1}}} \\ &= -\sum_{k=1}^{m-2}\left(\sum_{i=k+1}^{m-1} \frac{8^{i-k}b}{a^{m-i}} \frac{(i-1)\ldots(k+1)\,k}{(2i-1)\,(2i-3)\ldots(2k-1)} \frac{c^{i-k}}{\Delta^{i-k+1}}\right) \frac{(2cs+b)}{\sqrt{R^{2k-1}}}\end{aligned}$$

In addition to standard notations we accept the agreement that $0! = 1$, $(-1)!! = 1$. Then $\frac{(i-1)\ldots(k+1)k}{(2i-1)(2i-3)\ldots(2k-1)} = \frac{(i-1)!}{(k-1)!}\frac{(2k-3)!!}{(2i-1)!!}$. Therefore,

$$= -\sum_{k=1}^{m-2}\left(\sum_{i=k+1}^{m-1} \frac{8^{i-k}b}{a^{m-i}} \frac{(i-1)!}{(k-1)!} \frac{(2k-3)!!}{(2i-1)!!} \frac{c^{i-k}}{\Delta^{i-k+1}}\right) \frac{(2cs+b)}{\sqrt{R^{2k-1}}}$$

Returning to the calculated integral $\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}}$, we have:

$$\int \frac{\mathrm{d}s}{s\sqrt{R^{2m-1}}} = \sum_{k=1}^{m-1} \frac{1}{(2k-1)\,a^{m-k}\sqrt{R^{2k-1}}} - \frac{1}{\sqrt{a^{2m-1}}} \ln \frac{2a+bs+2\sqrt{aR}}{s} - \sum_{k=1}^{m-1} \frac{b}{2a^{m-k}} \frac{2\,(2cs+b)}{(2k-1)\,\Delta\sqrt{R^{2k-1}}} - \sum_{k=1}^{m-2} \left(\sum_{i=k+1}^{m-1} \frac{8^{i-k}b}{a^{m-i}} \frac{(i-1)!}{(k-1)!} \frac{(2k-3)!!}{(2i-1)!!} \frac{c^{i-k}}{\Delta^{i-k+1}} \right) \frac{(2cs+b)}{\sqrt{R^{2k-1}}}.$$

Using notations (15) and (16) in this indefinite integral, we have

$$\varepsilon_1(x,y) = \int_0^1 \left[\left| sx\,|y| - \frac{y}{|y|} \right|^{1-2m} - 1 \right] \frac{\mathrm{d}s}{s} = \left[\sum_{k=1}^{m-1} \frac{1}{(2k-1)\,a^{m-k}\sqrt{R^{2k-1}}} \right] \Bigg|_{s=0}^{s=1} + \left[-\frac{1}{\sqrt{a^{2m-1}}} \ln \frac{2a+bs+2\sqrt{aR}}{s} - \ln s \right] \Bigg|_{s=0}^{s=1} - \left[\sum_{k=1}^{m-1} \frac{b}{2a^{m-k}} \frac{2\,(2cs+b)}{(2k-1)\,\Delta\sqrt{R^{2k-1}}} - \sum_{k=1}^{m-2} \left(\sum_{i=k+1}^{m-1} \frac{8^{i-k}b}{a^{m-i}} \frac{(i-1)!}{(k-1)!} \frac{(2k-3)!!}{(2i-1)!!} \frac{c^{i-k}}{\Delta^{i-k+1}} \right) \frac{(2cs+b)}{\sqrt{R^{2k-1}}} \right] \Bigg|_{s=0}^{s=1},$$

we use the fact that

$$a = 1,\ 2a+2b+2\sqrt{aR\,(s)} = \begin{cases} 2+2=4, & \text{if } s=0 \\ 2-2\,(x,y)+2\left| x\,|y| - \frac{y}{|y|} \right|, & \text{if } s=1 \end{cases}$$

$$\sqrt{(R(s))^{2k-1}} = \begin{cases} 1, & \text{if } s=0, \\ \left| x\,|y| - \frac{y}{|y|} \right|^{2k-1}, & \text{if } s=1 \end{cases}$$

$$a=1,\ \frac{b}{2a^{m-k+1}} = -(x,y),\ 2cs+b = \begin{cases} b=-2\,(x,y), & \text{if } s=0 \\ 2c+b = 2\,|x|^2\,|y|^2 - 2\,(x,y), & \text{if } s=1 \end{cases},$$

$$\Delta = 4\left(|x|^2\,|y|^2 - (x,y)^2 \right),$$

$$c^{i-k} = |x|^{2i-2k}\,|y|^{2i-2k},\ \Delta^{i-k+1} = 4^{i-k+1}\left(|x|^2\,|y|^2 - (x,y)^2 \right)^{i-k+1}.$$

Then

$$\varepsilon_1(x,y) = \sum_{k=1}^{m-1} \frac{1}{(2k-1)} \left(\left| x\,|y| - \frac{y}{|y|} \right|^{1-2k} - 1 \right) + \ln \frac{4}{\left| 2-2\,(x,y)+2\left| x\,|y| - \frac{y}{|y|} \right| \right|} - \frac{-(x,y)}{4\left(|x|^2\,|y|^2 - (x,y)^2 \right)} \sum_{k=1}^{m-1} \frac{2}{(2k-1)} \left(\frac{2\,|x|^2\,|y|^2 - 2\,(x,y)}{\left| x\,|y| - \frac{y}{|y|} \right|^{2k-1}} - \frac{-2\,(x,y)}{1} \right)$$

$$+\sum_{k=1}^{m-2}\left(\frac{2|x|^2|y|^2-2(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}\right)\sum_{i=k+1}^{m-1}\frac{8^{i-k}(i-1)!(2k-3)!!}{4^{i-k+1}(k-1)!(2i-1)!!}$$
$$\times\frac{2(x,y)|x|^{2i-2k}|y|^{2i-2k}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i-k+1}}.$$

By reducing the constants and signs in the obtained formula, we find the representation for the function $\varepsilon_1(x,y)$:

$$\varepsilon_1(x,y)=\sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left(\left|x|y|-\frac{y}{|y|}\right|^{1-2k}-1\right)+\ln 2-\ln\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|$$
$$+\frac{(x,y)}{|x|^2|y|^2-(x,y)^2}\sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left(\frac{|x|^2|y|^2-(x,y)}{|x|y|-y/|y||^{2k-1}}+(x,y)\right)$$
$$+\sum_{k=1}^{m-2}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right)\sum_{i=k+1}^{m-1}2^{i-k}\frac{(i-1)!(2k-3)!!}{(k-1)!(2i-1)!!}$$
$$\times\frac{(x,y)|x|^{2i-k}|y|^{2i-k}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i-k+1}}.$$

We change the summation index in the last summand in the inner sum:

$$\sum_{i=k+1}^{m-1}\frac{(i-1)!(2k-3)!!}{(k-1)!(2i-1)!!}\frac{2^{i-k}|x|^{2i-k}|y|^{2i-k}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i-k+1}}=\left|\begin{array}{l}i-k=j, i=k+j\\ i=k+1\to j=1\\ i=m-1\to j=m-k-1\end{array}\right|$$
$$=\sum_{j=1}^{m-k-1}2^j\frac{(k+j-1)!}{(k-1)!}\frac{(2k-3)!!}{(2k+2j-1)!!}\frac{|x|^{2j}|y|^{2j}}{\left(|x|^2|y|^2-(x,y)^2\right)^{j+1}}=|i:=j|$$
$$=\sum_{i=1}^{m-k-1}2^i\frac{(k+i-1)!}{(k-1)!}\frac{(2k-3)!!}{(2k+2i-1)!!}\frac{|x|^{2i}|y|^{2i}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i+1}}.$$

Finally, we get

$$\varepsilon_1(x,y)=-\ln\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|+\ln 2+\sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left(\left|x|y|-\frac{y}{|y|}\right|^{1-2k}-1\right)$$
$$+\frac{(x,y)}{|x|^2|y|^2-(x,y)^2}\sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right)$$
$$+\sum_{k=1}^{m-2}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right)\sum_{i=1}^{m-k-1}\frac{(k+i-1)!(2k-3)!!}{(k-1)!(2k+2i-1)!!}$$
$$\times\frac{(x,y)2^i|x|^{2i}|y|^{2i}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i+1}}.$$

In such formula it is well seen that two last sums can be unite into one. The first sum is a case of the second one at $i = 0$ (except a summand with the number $k = m - 1$). Therefore, we finally have

$$\varepsilon_1(x, y) = -\ln\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right| + \ln 2 + \sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left(\left|x|y|-\frac{y}{|y|}\right|^{1-2k}-1\right)$$
$$+\frac{(x,y)}{|x|^2|y|^2-(x,y)^2}\frac{1}{(2m-3)}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2m-3}}+(x,y)\right)$$
$$+\sum_{k=1}^{m-2}\left(\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right)\sum_{i=0}^{m-k-1}\frac{(k+i-1)!\,(2k-3)!!}{(k-1)!\,(2k+2i-1)!!}\frac{(x,y)\,2^i|x|^{2i}|y|^{2i}}{\left(|x|^2|y|^2-(x,y)^2\right)^{i+1}}.$$

So, we have got the representation (iii) at $n \geq 5$, $n = 2m+1$, $m \geq 2$.

The fourth case: Now consider the remaining case $n \geq 6$ at the even n. We show that if $n \geq 6$, $n = 2m+2$, $m \geq 2$, then there is the representation (iv) for the function $\varepsilon_1(x, y)$. In this case for the function $\varepsilon_1(x, y)$, we get

$$\int_0^1\left[\left|sx|y|-\frac{y}{|y|}\right|^{-2m}-1\right]\frac{ds}{s} = \int_0^1\left[\frac{1}{\left(1-2s(x,y)+s^2|x|^2|y|^2\right)^m}-1\right]\frac{ds}{s}.$$

From results of the paper of Bitsadze [6] it follows

$$\int\frac{1}{(R(s))^m}\frac{ds}{s} = \frac{1}{2(m-1)(R(s))^{m-1}} + (x,y)\int\frac{ds}{(R(s))^m} + \int\frac{ds}{s(R(s))^{m-1}}.$$

Hence, by iteration we find:

$$\int\frac{1}{(R(s))^m}\frac{ds}{s} = \sum_{k=0}^{m-2}\left(\frac{1}{2(m-k-1)(R(s))^{m-k-1}}+(x,y)\int\frac{ds}{(R(s))^{m-k}}\right)+\int\frac{1}{R(s)}\frac{ds}{s}$$
$$= \sum_{k=0}^{m-2}\frac{1}{2(m-k-1)(R(s))^{m-k-1}}+(x,y)\sum_{k=0}^{m-2}\int\frac{ds}{(R(s))^{m-k}}+\int\frac{1}{R(s)}\frac{ds}{s},$$

changing here the order of summation in reverse direction, we get

$$\int\frac{1}{(R(s))^m}\frac{ds}{s} = \sum_{k=1}^{m-1}\frac{1}{2k(R(s))^k}+(x,y)\sum_{k=1}^{m-1}\int\frac{ds}{(R(s))^{k+1}}+\int\frac{1}{R(s)}\frac{ds}{s}.$$

We have calculated the last summand while considering the second case:

$$\int\frac{1}{R}\frac{ds}{s} = \frac{1}{2a}\ln\frac{s^2}{R} - \frac{b}{2a}\int\frac{ds}{R}.$$

Here we use the fact that $a = 1$, $b = -2(x, y)$, $\frac{-b}{2a} = (x, y)$. Then

$$\int\frac{1}{R}\frac{ds}{s} = \frac{1}{2}\ln\frac{s^2}{R} + (x,y)\int\frac{ds}{R}.$$

Therefore

$$\int \frac{1}{(R(s))^m}\frac{\mathrm{d}s}{s} = \sum_{k=1}^{m-1}\frac{1}{2k\,(R(s))^k} + (x,y)\sum_{k=1}^{m-1}\int \frac{\mathrm{d}s}{(R(s))^{k+1}} + (x,y)\int\frac{\mathrm{d}s}{R} + \frac{1}{2}\ln\frac{s^2}{R}.$$

Since (see [22, p.79, example 2.171, formula 1(96)a])

$$\int \frac{\mathrm{d}s}{(R(s))^{k+1}} = \frac{2cs+b}{2k+1}\sum_{i=0}^{k-1}\frac{2^i\,(2k+1)\dots(2k-2i+1)\,c^i}{k\dots(k-i)\,\Delta^{i+1}\,(R(s))^{k-i}} + 2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\int\frac{\mathrm{d}s}{R(s)},$$

then we will have:

$$\int \frac{1}{(R(s))^m}\frac{\mathrm{d}s}{s} = \sum_{k=1}^{m-1}\frac{1}{2k\,(R(s))^k} + (x,y)\sum_{k=1}^{m-1}\frac{2cs+b}{2k+1}\sum_{i=0}^{k-1}\frac{2^i\,(2k+1)\dots(2k-2i+1)\,c^i}{k\dots(k-i)\,\Delta^{i+1}\,(R(s))^{k-i}}$$
$$+\frac{1}{2}\ln\frac{s^2}{R} + (x,y)\left(\sum_{k=1}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\int\frac{\mathrm{d}s}{R(s)} + (x,y)\int\frac{\mathrm{d}s}{R},$$

if we take the notations $0! = 1$, $(-1)!! = 1$, then the last two summand can be unite into one:

$$(x,y)\left(\sum_{k=1}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\int\frac{\mathrm{d}s}{R(s)} + (x,y)\int\frac{\mathrm{d}s}{R(s)} = (x,y)\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\int\frac{\mathrm{d}s}{R(s)},$$

using, as in the proof of the second case (see [22, p.79, example 2.172, case $\Delta > 0$])

$$\int\frac{\mathrm{d}s}{R} = \frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}},$$

we get

$$(x,y)\left(\sum_{k=1}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\int\frac{\mathrm{d}s}{R(s)} + (x,y)\int\frac{\mathrm{d}s}{R(s)}$$
$$= (x,y)\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}}.$$

Finally we get:

$$\int \frac{1}{(R(s))^m}\frac{\mathrm{d}s}{s} = \sum_{k=1}^{m-1}\frac{1}{2k\,(R(s))^k} + (x,y)\sum_{k=1}^{m-1}\frac{2cs+b}{2k+1}\sum_{i=0}^{k-1}\frac{2^i\,(2k+1)\dots(2k-2i+1)\,c^i}{k\dots(k-i)\,\Delta^{i+1}\,(R(s))^{k-i}}$$
$$+\frac{1}{2}\ln\frac{s^2}{R} + (x,y)\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}}.$$

We transform the second sum. At first we change the index of summation:

$$\sum_{k=1}^{m-1}\frac{2cs+b}{2k+1}\sum_{i=0}^{k-1}\frac{2^i\,(2k+1)\,(2k-1)\ldots(2k-2i+1)\,c^i}{k\,(k-1)\ldots(k-i)\,\Delta^{i+1}\,(R\,(s))^{k-i}}=\left|\begin{array}{l}k-i=j,i=k-j\\ i=0\to j=k\\ i=k-1\to j=1\end{array}\right|$$
$$=\sum_{k=1}^{m-1}\frac{2cs+b}{2k+1}\sum_{j=1}^{k}\frac{2^{k-j}c^{k-j}}{\Delta^{k-j+1}\,(R\,(s))^j}\frac{(2k+1)\,(2k-1)\ldots(2j+3)\,(2j+1)}{k\,(k-1)\ldots(j+1)\,j}=|k\leftrightarrow j|$$
$$=\sum_{j=1}^{m-1}\frac{2cs+b}{2j+1}\sum_{k=1}^{j}\frac{2^{j-k}c^{j-k}}{\Delta^{j-k+1}\,(R\,(s))^k}\frac{(2j+1)\,(2j-1)\ldots(2k+3)\,(2k+1)}{j\,(j-1)\ldots(k+1)\,k}$$

and now we change the order of summation

$$=\sum_{k=1}^{m-1}\frac{2cs+b}{(R\,(s))^k}\left(\sum_{j=k}^{m-1}\frac{2^{j-k}c^{j-k}}{\Delta^{j-k+1}}\frac{1}{(2j+1)}\frac{(2j+1)(2j-1)\ldots(2k+3)\,(2k+1)}{j\,(j-1)\ldots(k+1)\,k}\right)$$

Here $(2j-1)$ remain in the numerator and denominator to make it clear what value $(2j+1)$ $(2j-1)\ldots(2k+3)\,(2k+1)$ has at $j=k$.

Finally we get a formula for the indefinite integral:

$$\int\frac{1}{(R\,(s))^m}\frac{\mathrm{d}s}{s}=\sum_{k=1}^{m-1}\frac{1}{2k\,(R\,(s))^k}$$
$$+(x,y)\sum_{k=1}^{m-1}\frac{2cs+b}{(R\,(s))^k}\left(\sum_{j=k}^{m-1}\frac{2^{j-k}c^{j-k}}{\Delta^{j-k+1}}\frac{1}{2j+1}\frac{(2j+1)\ldots(2k+3)\,(2k+1)}{j\,(j-1)\ldots(k+1)\,k}\right)$$
$$+\frac{1}{2}\ln\frac{s^2}{R}+(x,y)\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}}.$$

Now we directly pass to the calculation of the function $\varepsilon_1\,(x,y)$:

$$\varepsilon_1\,(x,y)=\int_0^1\left[\left|sx\,|y|-\frac{y}{|y|}\right|^{2-n}-1\right]\frac{\mathrm{d}s}{s}$$
$$=\int_0^1\left[\left|sx\,|y|-\frac{y}{|y|}\right|^{-2m}-1\right]\frac{\mathrm{d}s}{s}=\int_0^1\left[\frac{1}{(R\,(s))^m}-1\right]\frac{\mathrm{d}s}{s}$$
$$=\left[\sum_{k=1}^{m-1}\frac{1}{2k\,(R\,(s))^k}+\frac{1}{2}\ln\frac{s^2}{R}-\ln s+(x,y)\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cs}{\sqrt{\Delta}}\right]\Bigg|_{s=0}^{s=1}$$
$$+\left[(x,y)\sum_{k=1}^{m-1}\frac{2cs+b}{(R\,(s))^k}\left(\sum_{j=k}^{m-1}\frac{2^{j-k}c^{j-k}}{\Delta^{j-k+1}}\frac{1}{(2j+1)}\frac{(2j+1)(2j-1)\ldots(2k+3)\,(2k+1)}{j\,(j-1)\ldots(k+1)\,k}\right)\right]\Bigg|_{s=0}^{s=1},$$

we use the fact that

$$2cs+b=\begin{cases} b=-2\left(x,y\right), \text{ if } s=0 \\ 2c+b=2\left|x\right|^2\left|y\right|^2-2\left(x,y\right), \text{ if } s=1 \end{cases}, \Delta=4\left(\left|x\right|^2\left|y\right|^2-(x,y)^2\right),$$

$$\left(R(s)\right)^k=\begin{cases} 1, \text{ at } s=0, \\ \left|x\left|y\right|-\frac{y}{|y|}\right|^{2k}, \text{ at } s=1 \end{cases}$$

Then

$$\varepsilon_1\left(x,y\right)=\sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x\left|y\right|-\frac{y}{|y|}\right|^{-2k}-1\right)+\frac{1}{2}\ln\frac{1}{\left|x\left|y\right|-\frac{y}{|y|}\right|^2}$$

$$+\frac{(x,y)\left(\sum\limits_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)}{\sqrt{\left(\left|x\right|^2\left|y\right|^2-(x,y)^2\right)}}\left[\arctan\frac{\left|x\right|^2\left|y\right|^2-(x,y)}{\sqrt{\left|x\right|^2\left|y\right|^2-(x,y)^2}}-\arctan\frac{-(x,y)}{\sqrt{\left|x\right|^2\left|y\right|^2-(x,y)^2}}\right]$$

$$+(x,y)\sum\nolimits_{k=1}^{m-1}\left(\sum_{j=k}^{m-1}\frac{2^{j-k}c^{j-k}}{\Delta^{j-k+1}}\frac{(2j-1)\ldots(2k+1)}{j\ldots(k+1)\,k}\right)\left[\frac{2\left|x\right|^2\left|y\right|^2-2\left(x,y\right)}{\left|x\left|y\right|-\frac{y}{|y|}\right|^{2k}}-\frac{-2\left(x,y\right)}{1}\right].$$

We make elementary transformations, reduce the constants and signs, use the formula: $\arctan A-\arctan B=\arctan\frac{A-B}{1+AB}$. Then

$$\varepsilon_1\left(x,y\right)=\sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x\left|y\right|-\frac{y}{|y|}\right|^{-2k}-1\right)-\ln\left|x\left|y\right|-\frac{y}{|y|}\right|$$

$$+\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!c^k}{k!\Delta^k}\right)\frac{(x,y)}{\sqrt{\left|x\right|^2\left|y\right|^2-(x,y)^2}}\arctan\frac{\sqrt{\left|x\right|^2\left|y\right|^2-(x,y)^2}}{1-(x,y)}$$

$$+\sum_{k=1}^{m-1}\left(\sum_{j=k}^{m-1}\frac{2^{j-k+1}c^{j-k}}{\Delta^{j-k+1}}\frac{(x,y)}{(2j+1)}\frac{(2j+1)\ldots(2k+1)}{j\ldots(k+1)\,k}\right)\left[\frac{\left|x\right|^2\left|y\right|^2-(x,y)}{\left|x\left|y\right|-y/\left|y\right|\right|^{2k}}+(x,y)\right].$$

We substitute the value of parameters $c=\left|x\right|^2\left|y\right|^2$, $\Delta=4\left(\left|x\right|^2\left|y\right|^2-(x,y)^2\right)$ into the obtained formula. Then

$$\varepsilon_1\left(x,y\right)=\sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x\left|y\right|-\frac{y}{|y|}\right|^{-2k}-1\right)-\ln\left|x\left|y\right|-\frac{y}{|y|}\right|$$

$$+\left(\sum_{k=0}^{m-1}2^k\frac{(2k-1)!!}{k!4^k}\frac{\left|x\right|^{2k}\left|y\right|^{2k}}{\left(\left|x\right|^2\left|y\right|^2-(x,y)^2\right)^{k+\frac{1}{2}}}\right)\frac{(x,y)}{1}\arctan\frac{\sqrt{\left|x\right|^2\left|y\right|^2-(x,y)^2}}{1-(x,y)}$$

$$+\sum_{k=1}^{m-1}\left(\sum_{j=k}^{m-1}\frac{(x,y)\left(\frac{|x||y|}{\sqrt{2}}\right)^{2j-2k}}{2\left(\left|x\right|^2\left|y\right|^2-(x,y)^2\right)^{j-k+1}}\frac{(2j+1)\ldots(2k+1)}{(2j+1)j\ldots k}\right)\left[\frac{\left|x\right|^2\left|y\right|^2-(x,y)}{\left|x\left|y\right|-\frac{y}{|y|}\right|^{2k}}+(x,y)\right].$$

We reduce the constants:

$$\varepsilon_1(x,y)=\sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x\,|y|-\frac{y}{|y|}\right|^{-2k}-1\right)-\ln\left|x\,|y|-\frac{y}{|y|}\right|$$
$$+(x,y)\left(\sum_{k=0}^{m-1}2^{-k}\frac{(2k-1)!!}{k!}\frac{|x|^{2k}\,|y|^{2k}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{k+\frac{1}{2}}}\right)\arctan\frac{\sqrt{|x|^2\,|y|^2-(x,y)^2}}{1-(x,y)}$$
$$+\sum_{k=1}^{m-1}\left(\sum_{j=k}^{m-1}\frac{(x,y)\left(\frac{|x||y|}{\sqrt{2}}\right)^{2j-2k}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{j-k+1}}\frac{(2j+1)\ldots(2k+1)}{(2j+1)j\ldots k}\right)\left[\frac{|x|^2\,|y|^2-(x,y)}{\left|x\,|y|-\frac{y}{|y|}\right|^{2k}}+(x,y)\right].$$

We change the index of summation j into $i=j-k$ in the inner sum of the last summand. Then this inner sum will equal:

$$\sum_{j=k}^{m-1}\frac{2^{k-j-1}\,|x|^{2j-2k}\,|y|^{2j-2k}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{j-k+1}}\frac{(2j+1)(2j-1)\ldots(2k+3)\,(2k+1)}{(2j+1)\,j\,(j-1)\ldots(k+1)\,k}$$
$$=\sum_{i=0}^{m-k-1}2^{-i-1}\frac{|x|^{2i}\,|y|^{2i}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{i+1}}\frac{(2k+2i+1)(2k+2i-1)\ldots(2k+3)\,(2k+1)}{(k+i)\,(k+i-1)\ldots(k+1)\,k(2k+2i+1)}$$
$$=\sum_{i=0}^{m-k-1}2^{-i-1}\frac{|x|^{2i}\,|y|^{2i}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{i+1}}\frac{(2k+1)\,(2k+3)\ldots(2k+2i-1)(2k+2i+1)}{k(k+1)\ldots(k+i-1)\,(k+i)\,(2k+2i+1)}$$
$$=\sum_{i=0}^{m-k-1}2^{-i-1}\frac{|x|^{2i}\,|y|^{2i}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{i+1}}\frac{(2k+2i-1)!!(k+1)!}{(2k-1)!!\,(k+i)!}.$$

The last sum has the notation that $0!=1$, $(-1)!!=1$.

Finally we get:

$$\varepsilon_1(x,y)=\sum_{k=1}^{m-1}\frac{1}{2k}\left(\left|x\,|y|-\frac{y}{|y|}\right|^{-2k}-1\right)-\ln\left|x\,|y|-\frac{y}{|y|}\right|$$
$$+(x,y)\left(\sum_{k=0}^{m-1}2^{-k}\frac{(2k-1)!!}{k!}\frac{|x|^{2k}\,|y|^{2k}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{k+\frac{1}{2}}}\right)\arctan\frac{\sqrt{|x|^2\,|y|^2-(x,y)^2}}{1-(x,y)}$$
$$+\sum_{k=1}^{m-1}\left(\sum_{i=0}^{m-k-1}\frac{(x,y)\,2^{-i-1}|x|^{2i}\,|y|^{2i}}{\left(|x|^2\,|y|^2-(x,y)^2\right)^{i+1}}\frac{(2k+2i-1)!!(k+1)!}{(2k-1)!!\,(k+i)!}\right)\left[\frac{|x|^2\,|y|^2-(x,y)}{\left|x\,|y|-\frac{y}{|y|}\right|^{2k}}+(x,y)\right].$$

Thus we have got the representation (iv) of the function $\varepsilon_1(x,y)$ at $n\geq 6$, $n=2m+2$, $m\geq 2$, and it means that Lemma 2.2 is completely proved. □

3. The main results of the paper

THEOREM 3.1 *For the Green's function of the Neumann problem (5) we have the following representation*

$$G_N(x,y)=\frac{1}{\omega_n}\left[\varepsilon(x-y)+\varepsilon\left(x|y|-\frac{y}{|y|}\right)+\varepsilon_1(x,y)\right]+Const, \tag{17}$$

where the function $\varepsilon_1(x,y)$ *is given by* (14) *and is explicitly expressed in terms of elementary functions by formulas* (i)–(iv) *from Lemma* 2.2.

It is easy to see that at $n=3$ from (17) we obtain earlier known representation of the Green's function (10). And for $n\geq 4$ the result of the theorem is new.

Proof We will search the Green's function $G_N(x,y)$ in the form of (8), where $g(x,y)$ is an unknown harmonic function in D. We construct the function so that condition (9) holds.

Therefore the function $g(x,y)$ must satisfy the condition

$$\frac{\partial}{\partial n_y}g(x,y)+\frac{\partial}{\partial n_y}\varepsilon(x-y)=-1, \quad \text{for all } y\in\partial D. \tag{18}$$

We apply the method used in (see [2, p.348]) for constructing the Green's function of a Neumann three-dimensional problem. We search the function $g(x,y)$ in the form

$$g(x,y)=\sum_{k=0}^{\infty}b_k|x|^k|y|^k\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right), \tag{19}$$

where b_k are unknown coefficients.

Denote $r=|x|$, $\rho=|y|$. From (11) and (19) for any $r<\rho$, we calculate

$$\frac{\partial}{\partial\rho}\varepsilon(x-y)=\sum_{k=0}^{\infty}\frac{k+n-2}{2k+n-2}\frac{r^k}{\rho^{k+n-3}}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right),$$

$$\frac{\partial}{\partial\rho}g(x,y)=\sum_{k=0}^{\infty}b_k k r^k\rho^{k-1}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right).$$

Substituting the obtained formula into (18), for all indexes $k\geq 1$ we have:

$$b_k=\frac{1}{2k+n-2}+\frac{n-2}{k(2k+n-2)},\ k\geq 1.$$

The Green's function (17) is found up to an arbitrary constant. Therefore we can arbitrarily choose the coefficient b_0. We choose the coefficient $b_0=1/(n-2)$ for uniformity of further calculations.

Substituting the found coefficients into (19), we get

$$g(x,y)=\sum_{k=0}^{\infty}\frac{|x|^k|y|^k}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right)$$
$$+\sum_{k=1}^{\infty}\frac{(n-2)|x|^k|y|^k}{k(2k+n-2)}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right)\equiv g_1(x,y)+g_2(x,y).$$

The first sum gives the function

$$g_1(x,y)=\varepsilon\left(x|y|-\frac{y}{|y|}\right):$$

$$\varepsilon\left(x|y|-\frac{y}{|y|}\right)=\sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|x|y||^k}{|y/|y||^{k+n-2}}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x|y|}{|x|y||}\right)H_k^{(i)}\left(\frac{y/|y|}{|y/|y||}\right)$$

$$=\sum_{k=0}^{\infty}\frac{|x|^k|y|^k}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right)=g_1(x,y).$$

And for the second sum, using the equality $\frac{1}{k}=\int_0^1 s^{k-1}\mathrm{d}s$, $k\geq 1$, we get

$$g_2(x,y)=(n-2)\int_0^1\left[\sum_{k=1}^{\infty}\frac{s^k|x|^k|y|^k}{2k+n-2}\sum_{i=1}^{h_k}H_k^{(i)}\left(\frac{x}{|x|}\right)H_k^{(i)}\left(\frac{y}{|y|}\right)-\frac{1}{n-2}\right]\frac{\mathrm{d}s}{s},$$

hence, taking into account representation (11), we obtain (14). The theorem is proved. □

4. Concluding remarks

In conclusion we note that the solution of the Dirichlet problem (1) for the Laplace equation ($f\equiv 0$) is represented in the form of the Poisson integral

$$u(x)=\int_{\partial D}P(x,y)\,\varphi(y)\,\mathrm{d}s_y,$$

where $P(x,y)$ is a Poisson kernel found with the help of the Green's function of the Dirichlet problem by the formula

$$P(x,y)=-\frac{\partial}{\partial n_y}G_D(x,y)\quad \text{at } x\in D,\ y\in\partial D.$$

A similar representation holds for the solution of the Neumann problem (4) for the Laplace equation ($f\equiv 0$:)

$$u(x)=\frac{1}{\omega_n}\int_{\partial D}N(x,y)\,\psi(y)\,\mathrm{d}s_y,$$

where $N(x,y)$ is a Neumann kernel found with the help of the Green's function by the formula

$$N(x,y)=\omega_n G_N(x,y)\quad \text{at } x\in D,\ y\in\partial D.$$

The method of constructing the explicit form of the function $N(x,y)$ for a multi-dimensional unit sphere was considered in the paper of Bitsadze [6]. It was shown that the Neumann kernel could be expressed in elementary functions. The explicit form of the Neumann kernel at $n=4$ was given.

From the representation (17) of the Green's function and the representation (ii) of Lemma 2.2 in case $n=4$, assuming $|y|=1$, we get

$$N(x,y)=|x-y|^{-2}-\ln|x-y|+\frac{(x,y)}{\sqrt{|x|^2-(x,y)^2}}\arctan\frac{\sqrt{|x|^2-(x,y)^2}}{1-(x,y)}.$$

This equality fully coincides with (21) from [6].

We demonstrate the form for the Neumann kernel $N(x, y)$ at $n = 5$, obtained from (17) and the representation (iii). From (17) at $|y| = 1$, we obtain

$$\begin{aligned} N(x, y) &= \varepsilon(x - y) + \varepsilon(x - y) + \varepsilon_1(x, y) = 2\varepsilon(x - y) + \varepsilon_1(x, y) \\ &= \frac{2}{3}\frac{1}{|x - y|^3} + \frac{1}{|x - y|} + \frac{(x, y)}{|x|^2 - (x, y)^2}\left(\frac{|x|^2 - (x, y)}{(x, y)} + (x, y)\right) \\ &\quad - \ln|1 - (x, y) + |x - y||. \end{aligned}$$

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