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Unique solvability of boundary value problem for functional differential equations with involution

In this paper we consider a boundary value problem for systems of Fredholm type integral-differential equations with involutive transformation, containing derivative of the required function on the right-hand side under the integral sign. Applying properties of an involutive transformation, original boundary value problem is reduced to a boundary value problem for systems of integral-differential equations, containing derivative of the required function on the right side under the integral sign. Assuming existence of resolvent of the integral equation with respect to the kernel $\tilde{K}_2(t,s)$ (this is the kernel of the integral equation that contains the derivative of the desired function) and using properties of the resolvent, integral-differential equation with a derivative on the right-hand side is reduced to a Fredholm type integral-differential equation, in which there is no derivative of the desired function on the right side of the equation. Further, the obtained boundary value problem is solved by the parametrization method created by Professor D. Dzhumabaev. Based on this method, the problem is reduced to solving a special Cauchy problem with respect to the introduced new functions and to solving systems of linear algebraic equations with respect to the introduced parameters. An algorithm to find a solution is proposed. As is known, in contrast to the Cauchy problem for ordinary differential equations, the special Cauchy problem for systems of integral-differential equations is not always solvable. Necessary conditions for unique solvability of the special Cauchy problem were established. By using results obtained by Professor D. Dzhumabaev, necessary and sufficient conditions for the unique solvability of the original problem were established.

Keywords: System of integral-differential equations, boundary value conditions, parametrization method, integral equation, resolvent, involution, unique solvability, Special Cauchy Problem.

Introduction

Boundary value problems for integral-differential equations have been studied by many authors [1–7], however, with the development of computer technology, the question of creating constructive methods for solving the problem arises. In connection with this, Professor D. Dzhumabaev proposed a method for parameterizing the solution of a linear two-point boundary value problem for systems of differential equations [8]. This method was applied to study various boundary value problems [9–14].

On the segment $[0,T]$ we consider the following boundary value problem:

$$\frac{dx(t)}{dt} + \mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(\alpha(t))}{dt} = \int_0^T K_1(t,s)x\,(s)\,ds + \int_0^T K_2(t,s)\dot{x}\,(s)\,ds + f(t), \quad t\in[0,T]\,, \tag{1}$$

$$Bx(0) + C(T) = d, d\in R^n, \tag{2}$$

where the matrices $K_1(t,s)$, $K_2(t,s)$ are continuous on $[0,T]\times[0,T]$, respectively, n-dimensional vector-function $f(t)$ is continuous on $[0,T]$. $\alpha(t)$ is a reorientation homeomorphism $\alpha:[0,T]\to[0,T]$ such that $\alpha^2(t) = \alpha(\alpha(t)) = t$. It is known that the homeomorphism $\alpha(t)$ is called the involutive transformation. On the segment $[0,T]$ as such a transformation, we can consider the transformation $\alpha(t) = T-t$. Properties of the involutive transformation were studied by G.S. Litvinchuk [14], N.K. Karapetyants and S.G. Samko [15] and others.

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We consider a value of equation (1) at the point $t=\alpha(t)$

$$\frac{dx(\alpha(t))}{dt}+\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(t)}{dt}=\int\limits_0^T K_1(\alpha(t),s)x\left(s\right)ds+\int\limits_0^T K_2(\alpha(t),s)\dot{x}\left(s\right)ds+f(\alpha(t)).$$

From the system

$$\begin{cases} \frac{dx(t)}{dt}+\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(\alpha(t))}{dt}=\int\limits_0^T K_1(t,s)x\left(s\right)ds+\int\limits_0^T K_2(t,s)\dot{x}\left(s\right)ds+f(t), \\ \frac{dx(\alpha(t))}{dt}+\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)\frac{dx(t)}{dt}=\int\limits_0^T K_1(\alpha(t),s)x\left(s\right)ds+\int\limits_0^T K_2(\alpha(t),s)\dot{x}\left(s\right)ds+f(\alpha(t)) \end{cases}$$

we define

$$\mathrm{diag}(1-a_1^2,1-\ a_2^2,\ \ldots,1-\ a_n^2)\frac{dx(t)}{dt}=\int\limits_0^T\left[K_1(t,s)-\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_1(\alpha(t),s)\right]x(s)ds+$$

$$+\int\limits_0^T\left[K_2(t,s)-\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_2(\alpha(t),s)\right]\dot{x}(s)ds+\left[f(t)-\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)f(\alpha(t))\right].$$

Suppose that the matrix $\mathrm{diag}(1-a_1^2,1-\ a_2^2,\ \ldots,1-\ a_n^2)$ is not degenerate, then it is invertible, and boundary value problem (1)–(2) can be written in the form

$$\frac{dx}{dt}=\int\limits_0^T\tilde{K}_1(t,s)x(s)ds+\int\limits_0^T\tilde{K}_2(t,s)\dot{x}(s)ds+\tilde{f}(t),\quad t\in[0,T]\,,\tag{3}$$

$$Bx(0)+Cx(T)=d,\ \ d\in R^n,\tag{4}$$

where

$$\tilde{K}_1(t,s)=\mathrm{diag}(1/(1-a_1^2),\ 1/(1-a_2^2),\ldots,1/(1-a_n^2))\left[K_1(t,s)-\mathrm{diag}(a_1,\ a_2,\ \ldots,a_n)K_1(\alpha(t),s)\right],$$

$$\tilde{K}_2(t,s)=\mathrm{diag}(1/(1-a_1^2),\ 1/(1-a_2^2),\ldots,\ 1/(1-a_n^2))\left[K_2(t,s)-\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)K_2(\alpha(t),s)\right],$$

$$\tilde{f}(t)=\mathrm{diag}(1/(1-a_1^2),\ 1/(1-a_2^2),\ldots,\ 1/(1-a_n^2))\left[f(t)-\mathrm{diag}(a_1,\ a_2,\ \ldots,\ a_n)f(\alpha(t))\right].$$

Condition A. Let the following Fredholm integral equation of the second kind

$$z(t)=\int\limits_0^T\tilde{K}_2(t,s)z\left(s\right)ds+\Phi(t)$$

has a unique solution for any function $\Phi(t)\in C\left([0,T]\,,R^n\right)$.

If Condition A holds, then there exists $\Gamma_2(t,s;1)$ – resolvent of the Fredholm integral equation of the second kind with the kernel $\tilde{K}_1(t,s)$ and a solution of the integral equation can be written as

$$z^*(t)=\Phi(t)+\int\limits_0^T\Gamma_2(t,s;1)\Phi\left(s\right)ds.$$

By using Condition A, problem (3) – (4) can be rewritten as

$$\frac{dx}{dt}=\int\limits_0^T\tilde{K}_1(t,s)x(s)ds+\tilde{f}(t)+\int\limits_0^T\Gamma_2(t,\tau;1)\left[\int\limits_0^T\tilde{K}_1(\tau,s)x(s)ds+\tilde{f}(\tau)\right]d\tau,\quad t\in[0,T]\,,\tag{5}$$

$$Bx(0)+Cx(T)=d,\ \ d\in R^n.\tag{6}$$

Changing the order of integration in the integral term we obtain

$$\int_0^T \Gamma_2(t,s;1)\int_0^T \tilde{K}_1(s,\tau)x(\tau)d\tau ds = \int_0^T \left(\int_0^T \Gamma_2(t,\tau;1)\tilde{K}_1(\tau,s)d\tau\right) x(s)ds = \int_0^T K^*{}_1(t,s)x(s)ds.$$

We denote

$$\hat{K}_1(t,s) = K^*{}_1(t,s) + \tilde{K}_1(t,s),$$

$$\hat{f}(t) = \tilde{f}(t) + \int_0^T \Gamma_2(t,\tau;1)\tilde{f}(\tau)d\tau.$$

Then we rewrite problem (5) – (6) in the form:

$$\frac{dx}{dt} = \int_0^T \hat{K}_1 t,s)x(s)ds + \hat{f}(t), \quad t \in [0,T], \tag{7}$$

$$Bx(0) + Cx(T) = d, \ \ d \in R^n. \tag{8}$$

We take the step $h > 0$, that fits N times on the segment $[0,T]$ and along it we consider the partition $[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh)$.

We denote restriction of the function $x(t)$ on the r-th interval $[(r-1)h, rh)$ by $x_r(t)$, i.e., $x_r(t)$ is a system of vector functions defined and coinciding with $x(t)$ on $[(r-1)h, rh)$. Then the original two-point boundary value problem for systems of integral-differential equations is reduced to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} \hat{K}_1(t,s)x_j(s)ds + \hat{f}(t), t \in [(r-1)h, rh), \tag{9}$$

$$Bx_1(0) + C\lim_{t\to T-0} x_N(t) = d, \tag{10}$$

$$\lim_{t\to sh-0} x_s(t) = x_{s+1}(sh), s = \overline{1, N-1}. \tag{11}$$

Here (11) are gluing conditions at the interior points of the partition $t = jh$, $j = \overline{1, N-1}$.

If the function $x(t)$ is a solution to problem (7)–(8), then the system of its restrictions $x[t] = (x_1(t), x_2(t), ..., x_N(t))'$ will be a solution of multipoint boundary value problem (9)–(11). And in inverse, if the system of vector functions $\tilde{x}[t] = (\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_N(t))'$ is a solution to problem (9)–(11), then the function $\tilde{x}(t)$, defined by the equalities $\tilde{x}(t) = \tilde{x}_r(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1, N}$, $\tilde{x}(T) = \lim_{t\to T-0} \tilde{x}_N(t)$ will be a solution of original boundary value problem (7)–(8). By λ_r we denote a value of the function $x_r(t)$ at the point $t = (r-1)h$ and on each interval $[(r-1)h, rh)$ we change $x_r(t) = u_r(t) + \lambda_r$, $r = \overline{1, N}$. Then problem (9)–(11) is reduced to the equivalent multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} \hat{K}_1(t,s)[u_j(s) + \lambda_j]ds + \hat{f}(t), \tag{12}$$

$$u_r[(r-1)h] = 0, t \in [(r-1)h, rh), r = \overline{1, N}, \tag{13}$$

$$B\lambda_1 + C\lambda_N + C\lim_{t\to T-0} u_N(t) = d, \tag{14}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1}, s = \overline{1, N-1}. \tag{15}$$

Problems (9)–(11) and (12)–(15) are equivalent in the sense that if the system of functions $x[t] = (x_1(t), x_2(t), ..., x_N(t))'$ is a solution of problem (9)–(11), then pair $(\lambda, u[t])$ will be a solution of the problem (12)–(15), where $\lambda = (x_1(0), x_2(h), ..., x_N(N-1)h))'$, $u[t] = (x_1(t) - x_1(0), x_2(t) - x_2(h), ..., x_N(t) - x_N((N-1)h))$. And in inverse, if pair $(\tilde{\lambda}, \tilde{u}[t])$ is a solution of the problem (12)–(15), where $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_N)'$,

$\tilde{u}[t] = (\tilde{u}_1(t), \tilde{u}_2(t), ..., \tilde{u}_N(t))'$, then the system of functions $\tilde{x}[t] = (\tilde{\lambda}_1 + \tilde{u}_1(t), \tilde{\lambda}_2 + \tilde{u}_2(t), ..., \tilde{\lambda}_N + \tilde{u}_N(t))'$ will be a solution of problem (9)–(11).

Appearance of the initial conditions $u_r[(r-1)h] = 0, r = \overline{1, N}$, allows us to determine functions $u_r(t)$, $r = \overline{1, N}$, from the systems of integral equations for fixed values $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$:

$$u_r(t) = \int\limits_{(r-1)h}^{t} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau, s)\,[u_j(s) + \lambda_j]\,ds d\tau + \int\limits_{(r-1)h}^{t} \hat{f}(\tau) d\tau, \qquad t \in [(r-1)h, rh). \tag{16}$$

From (16) defining $\lim\limits_{t \to Nh-0} u_N(t)$, $\lim\limits_{t \to sh-0} u_s(t)$, $s = \overline{1, N} - 1$, putting the corresponding expressions into the conditions (14), (15), and multiplying both sides of (14) to $h > 0$, we get the system of linear equations concerning to the unknown parameters λ_r, $r = \overline{1, N}$:

$$hB\lambda_1 + hC\lambda_N + hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau, s)\lambda_j ds d\tau =$$

$$= hd - hC \int\limits_{(N-1)h}^{Nh} \hat{f}(\tau)(\tau) d\tau - hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau, s) u_j(s) ds d\tau \tag{17}$$

$$\lambda_s + \int\limits_{(s-1)h}^{sh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau, s)\lambda_j ds\, d\tau - \lambda_{s+1} =$$

$$= - \int\limits_{(s-1)h}^{sh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} \hat{K}_1(\tau, s)\, u_j\,(s)\, ds\, d\tau - \int\limits_{(s-1)h}^{sh} \hat{f}(\tau) d\tau, \qquad s = \overline{1, N-1.} \tag{18}$$

We denote the $nN \times nN$ dimensional matrix corresponding to the left side of the system of linear equations (17), (18) by $Q(h)$. Then the system of linear equations (17), (18) can be written in the form:

$$Q(h)\lambda = -F(h) - G(u, h), \lambda \in R^{nN}, \tag{19}$$

where

$$F(h) = \left(-hd + hC \int\limits_{(N-1)h}^{Nh} f_1(\tau) d\tau,\ \int\limits_{0}^{h} f_1(\tau) d\tau,\ \ldots,\ \int\limits_{(N-2)h}^{(N-1)h} f_1(\tau) d\tau \right),$$

$$G(u, h) = \left(hC \int\limits_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_1(\tau, s) u_j(s) ds d\tau\ ,\ \int\limits_{0}^{h} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K_2(\tau, s) u_j(s) ds d\tau, \ldots, \right.$$

$$\left. \int\limits_{(N-2)h}^{(N-1)h} \sum_{j=1}^{N} \int\limits_{(j-1)h}^{jh} K(\tau, s) u_j(s) ds d\tau \right).$$

Therefore, to find unknown pairs $(\lambda, u[t])$, solutions of the problem (12)–(15)... we have a closed system of equations (16), (19). We find solution of the multipoint boundary value problem (12)–(15) as a limit of the sequence of pairs $(\lambda'^{(k)}, u^{(k)}[t])$, k =0,1,2, $\ldots$, defined by the following algorithm:

Step 0. a) Assuming, that the matrix $Q(h)$ is invertible, from the equation $Q(h)\lambda = -F(h)$ we define the initial approximation by the parameter $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}) \in R^{nN}$: $\lambda^{(0)} = -[Q(h)]^{-1}F(h)$.

b) Putting the found $\lambda_r^{(0)}$, $r = \overline{1, N}$ into the right side of the system of integral-differential equations (12) and solving the special Cauchy problem with conditions (13), we find $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), \ldots, u_N^{(0)}(t))'$.

Step 1. a) Putting the found values $u_r^{(0)}(t)$, $r = \overline{1, N}$ into the right side of (19), from the equation $[Q(h)]\lambda = -F(h) - G(u^{(0)}, h)$ we define $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots \lambda_N^{(1)})$.

b) Putting the found $\lambda_r^{(1)}$, $r = \overline{1, N}$ into the right side of the system of integral-differential equations (12) and solving the special Cauchy problem with conditions (13), we find $u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), \ldots, u_N^{(1)}(t))'$ and etc.

Continuing the process, at the k-step of the algorithm we find the system of pairs $(\lambda^{(k)}, u^{(k)}[t])$, k =0,1,2, ...

Unknown functions $u[t] = (u_1(t), u_2(t), \ldots, u_N(t))$ are determined from the special Cauchy problem for systems of integral-differential equations (12) with initial conditions (13). In contrast to the Cauchy problem for ordinary differential equations, the special Cauchy problem for systems of integral-differential equations is not always solvable.

Sufficient conditions for unique solvability of the special Cauchy problem (12), (13) for known values of the parameters λ are established by

Theorem 1. Let the partition step $h = T/N$ satisfy the inequality

$$\delta(h) = \beta T h < 1,$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} ||\hat{K}_1(t,s)||$.

Then the special Cauchy problem (12), (13) has a unique solution.

Sufficient conditions for feasibility and convergence of the proposed algorithm, as well as existence of a unique solution to problem (1), (2) are established by

Theorem 2. Let the following conditions hold:

1) Condition A,

2) matrix $\operatorname{diag}(1 - a_1^2, 1 - a_2^2, \ldots, 1 - a_n^2)$ is invertible,

3) conditions of Theorem 1 hold,

4) matrix $Q(h)$ is invertible and the following inequalities hold:

$$||[Q(h)]^{-1}|| \le \gamma(h),$$

$$q(h) = \frac{\delta(h)}{1 - \delta(h)} \gamma(h) \max(1, h\, \|C\|) \delta(h) < 1.$$

Then the two-point boundary value problem for systems of integral-differential equations (1), (2) has a unique solution.

Proof of Theorem 1 and Theorem 2 is similar to the scheme of the proof of Theorem 1 and Theorem 3 from [16] and is carried out according to the above algorithm, taking into account the specifics of the system (1).

In [5], necessary and sufficient conditions for unique solvability of a linear boundary value problem for the following systems of differential equations were obtained

$$\frac{dx}{dt} = \int\limits_0^T K(t,s)x(s)ds + f(t), \quad t \in [0,T],$$

$$Bx(0) + Cx(T) = d, \;\; d \in R^n.$$

Theorem ([8; 1216]). For unique solvability of the problem (14), (15) it is necessary and sufficient existence of $h \in (0, h_0] : Nh = T$, where the matrix $Q_{*,*}(h)$ is invertible.

The above theorem implies

Corollary. For unique solvability of the problem (1), (2) it is necessary and sufficient the conditions 1 and 2 of Theorem 2, as well as existence of $h \in (0, h_0] : Nh = T$, where the matrix $Q_{*,*}(h)$ is invertible.

Where h_0 is defined from the condition $q(h_0) = \beta T h_0 < 1$, and the matrix $Q_{*,*}(h)$ is defined in the same way as in [8].

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Инволюциялы функционалды-дифференциалдық теңдеулер үшін шеттік есептің бірмәнді шешілімділігі

Мақалада теңдеудің оң жағының құрамында интеграл таңбасының астында ізделінді функциядан туындысы бар инволютивті түрлендірумен Фредгольм типтес интегралдық-дифференциалдық теңдеулер жүйесі үшін шеттік есеп қарастырылды. Инволютивті түрлендірудің қасиетін пайдаланудан бастапқы есеп оң жақ бөлігінде интеграл таңбасының астында ізделінді функциядан туындысы бар интегралдық-дифференциалдық теңдеу үшін шеттік есепке және интегралдық теңдеудің ядросы $\tilde{K}_2(t,s)$-ке (ізделінді функциядан туындысы бар интегралдық теңдеудің ядросы) байланысты резольвентасы бар деп жорамалдап, интегралдық-дифференциалдық теңдеу оң жақ бөлігінде ізделінді функциядан туындысы жоқ теңдеуге келтіріледі. Алынған шеттік есеп профессор Д.С. Джумабаев ұсынған параметрлеу әдісімен шығарылған. Осы әдістің негізінде есеп жаңа енгізілген функцияларға байланысты арнайы Коши есебін және енгізілген параметрлерге байланысты сызықты алгебралық теңдеулер жүйесі шешуге келтіріледі. Есептің шешімін табу алгоритмі ұсынылған. Белгілі болғандай, жәй дифференциалдық теңдеулер үшін Коши есебіне қарағанда интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебінің барлық уақытта шешімі бар бола бермейді. Профессор Д.С. Джумабаевтың алған нәтижелерін қолдана отырып, арнайы Коши есебінің бірмәнді шешілімділігінің қажетті шарттары тағайындалды.

Кілт сөздер: интегралдық-дифференциалдық теңдеулер жүйесі, шеттік шарттар, параметрлеу әдісі, интегралдық теңдеу, резольвента, инволюция, бірмәнді шешілімділік, арнайы Коши есебі.

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Однозначная разрешимость краевой задачи для функционально-дифференциальных уравнений с инволюцией

В статье рассмотрена краевая задача для систем интегро-дифференциальных уравнений типа Фредгольма с инволютивным преобразованием, содержащая в правой части производную от искомой функций под знаком интеграла. Пользуясь свойством инволютивного преобразования, задача сведена к краевой задаче для систем интегро-дифференциальных уравнений, содержащей в правой части производную от искомой функции под знаком интеграла. Предполагая существование резольвенты интегрального уравнения относительно ядра $\tilde{K}_2(t,s)$ (ядро интегрального уравнения, которое содержит производную от искомой функции) и используя резольвенту, интегро-дифференциальное уравнение сведено к уравнению, не содержащему производную от искомой функции в правой части интегро-дифференциального уравнения. Далее полученная краевая задача решается методом параметризации, предложенным профессором Д. Джумабаевым. На основе данного метода задача сведена к решению специальной задачи Коши относительно введенных новых функций и к решению систем линейных алгебраических уравнений относительно введенных параметров. Предложен алгоритм нахождения решений. Как известно, в отличие от задачи Коши для обыкновенных дифференциальных уравнений, специальная задача Коши для систем интегро-дифференциальных уравнений не всегда разрешима. Авторами были установлены необходимые условия однозначной разрешимости специальной задачи Коши. Используя результаты, полученные профессором Д. Джумабаевым, были найдены необходимые и достаточные условия однозначной разрешимости исходной задачи.

Ключевые слова: система интегро-дифференциальных уравнений, краевые условия, метод параметризации, интегральное уравнение, резольвента, инволюция, однозначная разрешимость, специальная задача Коши.

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