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An Inverse Problem for a Parabolic Equation with Involution

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Abstract—In this paper, we investigate the solvability of an inverse problem with the Dirichlet condition for a nonlocal analog of a parabolic equation, which generalizes the classical parabolic equation in two spatial variables. Theorems on existence and uniqueness of solution to the considered problem are proved. The inverse problem is solved by using the variable separation method. To solve this problem, we use the method of separation of variables, the application of which leads to the study of the spectral problem for the Laplace equation with involution. Also, we have shown the method for reducing this problem to the known spectral problem for the classical Laplace equation (without involution). The completeness and basis property of the eigenfunctions of the obtained spectral problem are proved. This made it possible to represent the solution of the inverse problem in the form of a sum of an absolutely and uniformly converging series, expanded in terms of the obtained eigenfunctions.

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1. INTRODUCTION

Concept of a nonlocal operator and the related concept of a nonlocal differential equation appeared in mathematics relatively recently. According to the classification given by A. M. Nakhushev in the book [1], such equations include: loaded equations; equations containing fractional derivatives of the desired function; equations with deviation arguments, in other words, equations in which the unknown function and its derivatives enter, generally speaking, for different values of arguments. Among nonlocal differential equations, a special place is occupied the equations in which deviation of arguments has an involutive character. A mapping S is usually called involution, if $S^2 = E$, E is the identity mapping. In this paper, we introduce concept of a nonlocal Laplace operator and investigate spectral questions of some boundary value problems. Questions on solvability of some boundary value problems for a fractional analogue of the nonlocal Laplace equation will be studied in a three-dimensional parallelepiped.

Differential equations with involution have been studied in works of numerous authors [2–10]. In the work of A. V. Linkov [10] boundary value and initial - boundary value problems are investigated in the domain $(t, x) : t > 0, -\pi < x < \pi$ for analogues of a parabolic, hyperbolic and elliptic equation with involution

$$u_t(t,x) - u_{xx}(t,x) - \varepsilon u_{xx}(t,-x) = 0, \quad t > 0, \quad -\pi < x < \pi,$$
$$u_{tt}(t,x) - u_{xx}(t,x) - \varepsilon u_{xx}(t,-x) = 0, \quad t > 0, \quad -\pi < x < \pi,$$
$$u_{tt}(t,x) + u_{xx}(t,x) + \varepsilon u_{xx}(t,-x) = 0, \quad t > 0, \quad -\pi < x < \pi.$$

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Application of the Fourier method to these problems leads to the one-dimensional spectral problem

$$y''(x) + \varepsilon y''(x) = -\lambda y(x), \quad -\pi < x < \pi$$

with the corresponding boundary value conditions. Eigenfunctions of this problem will be the functions

$$y_{k,1}(t) = \sin kt, \quad y_{k,1}(t) = \cos(k + 0.5)t,$$

and eigenvalues are

$$\lambda_k^{(1)} = (1 - \varepsilon)k, \quad \lambda_k^{(2)} = (1 + \varepsilon)(k + 0.5).$$

Moreover, it should be noted here that the eigenfunctions of the equation with involution coincide with the eigenfunctions of the classical equation, i.e. $\varepsilon = 0$, and difference in these problems will be only in eigenvalues. In this paper, we consider two-dimensional generalization of an analog of parabolic equation. In the paper, by using the Fourier method, we study solvability of inverse problems with the Dirichlet condition. Let us turn to statement of the problem. Let $0 < p, q, T$ be real numbers, $\Pi = \{x = (x_1, x_2) \in R^2 : 0 < x_1 < p, 0 < x_2 < q\}$ be a rectangle, $Q = (0, T) \times \Pi$. For a point $x = (x_1, x_2) \in \Pi$ we consider the mapping

$$S_0 x = (x_1, x_2), \quad S_1 x = (p - x_1, x_2), \quad S_2 x = (x_1,\, q - x_2), \quad S_3 x = (p - x_1,\, q - x_2).$$

It is obvious that for any $j = \overline{0,3}$ we get $S_j^2 x = x$, i.E. mappings S_j are involution. Moreover, we have

$$S_1 \cdot S_2 = S_2 \cdot S_1 = S_3, \quad S_1 \cdot S_3 = S_3 \cdot S_1 = S_2, \quad S_2 \cdot S_3 = S_3 \cdot S_2 = S_1.$$

Let a_j be real numbers, $j = \overline{0,3}$, Δ be a Laplace operator, acting by the variables x_1 and x_2. For a function $v(x_1, x_2) \in C^2(\Pi)$ we consider the operator:

$$Lv(x) \equiv a_0 \Delta v(S_0 x) + a_1 \Delta v(S_1 x) + a_2 \Delta v(S_2 x) + a_3 \Delta v(S_3 x).$$

We call the operator L as the nonlocal Laplace operator. If $a_0 = 1, a_j = 0, j = 1, 2, 3,$ then L coincides with the usual two-dimensional Laplace operator. In the domain Q consider the following equation

$$\frac{\partial u(t,x)}{\partial t} = a_0 \Delta u(t, S_0 x) + a_1 \Delta u(t, S_1 x) + a_2 \Delta u(t, S_2 x) + a_3 \Delta u(t, S_3 x) + f(x), \quad (t,x) \in Q. \quad (1)$$

Here a_j are real numbers, $\Delta u(t, S_j x)$ means $\Delta u(t, S_j x) = \Delta u(t,z)|_{z=S_j x}$, $j = \overline{0,3}$. If $a_0 = 1$, $a_j = 0$, $j = 1, 2, 3$, then (1) coincides with the classical parabolic equation.

In the domain Q consider the following problem.

Problem ID. Find a solution of the equation (1) in the domain Q, satisfying the conditions

$$u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad x \in \bar{\Omega}_x, \quad (2)$$

$$u(t,x) = 0, \quad (t,x) \in \partial\Pi \times [0,T]. \quad (3)$$

Here $\varphi(x)$, $\psi(x)$ are given functions on Π, moreover, the following conditions of agreement hold:

$$\varphi(x)\,|_{\partial\Pi} = 0, \quad \psi(x)\,|_{\partial\Pi} = 0. \quad (4)$$

By a regular solution to problem ID we mean a pair of functions $(u(x,t), f(x))$ such that $u(t,x) \in C_{t,x}^{1,2}(\bar{Q})$,$f(x) \in C(\bar{\Pi})$, satisfying equation (1) and conditions (2)–(3) in the classical sense in the domain Q. Note that direct and inverse problems for a parabolic type equation of integer and fractional orders in the one-dimensional case were studied in [12–21], and in the two-dimensional case in [22, 23].

2. ON EIGENFUNCTIONS AND EIGENVALUES OF THE DIRICHLET BOUNDARY VALUE PROBLEM FOR THE NONLOCAL LAPLACE EQUATION

We consider in Π the following spectral problem.

Problem (Eigenfunctions and Eigenvalues of the Problem D). Find a function $v(x)\neq 0$ and a number $\lambda\in C$ satisfying the conditions:

$$-Lv(x)=\lambda v(x),\quad x\in\Pi, \tag{5}$$

$$\begin{aligned} v(x_1,0)=v(x_1,q)=0,\quad 0\le x_1\le p,\\ v(0,x_2)=v(p,x_2)=0,\quad 0\le x_2\le q. \end{aligned} \tag{6}$$

First, we present a well-known statement concerning the following spectral problem

$$-\Delta w(x_1,x_2)=\mu w(x_1,x_2),\quad (x_1,x_2)\in\Pi, \tag{7}$$

$$w(x_1,0)=w(x_1,q)=0,\quad 0\le x_1\le p,\quad w(0,x_2)=w(p,x_2)=0,\quad 0\le x_2\le q. \tag{8}$$

Let

$$X_k(x_1)=\sqrt{\frac{2}{p}}\sin\frac{k\pi}{p}x_1,\quad Y_m(x_2)=\sqrt{\frac{2}{q}}\sin\frac{m\pi}{q}x_2,\quad k,m=1,2,...,$$
$$\nu_k=\left(\frac{k\pi}{p}\right)^2,\quad \sigma_m=\left(\frac{m\pi}{q}\right)^2,\quad k,m=1,2,....$$

The following statement is well known (see e.g. [24]).

Lemma 1. *Eigenfunctions and eigenvalues of problem (7), (8) are:*

$$w_{k,m}(x_1,x_2)=X_k(x_1)Y_m(x_2)\equiv\sqrt{\frac{2}{p}}\sin\frac{k\pi}{p}x_1\sqrt{\frac{2}{q}}\sin\frac{m\pi}{q}x_2,$$
$$\mu_{k,m}=\nu_k+\sigma_m\equiv\left(\frac{k\pi}{p}\right)^2+\left(\frac{m\pi}{q}\right)^2,\quad k,m=1,2,...$$

System of the functions $\{w_{k,m}(x_1,x_2)\}_{k,m=1}^{\infty}$ *forms a complete orthonormal system in the space* $L_2(\Pi)$.

Let $w(x)$ be an eigenfunction of the problem (7), (8). Consider the following functions

$$w_1^{\pm}(x)=\frac{w(x)\pm w(S_1x)}{2};\quad w_2^{\pm}(x)=\frac{w(S_2x)\pm w(S_3x)}{2}.$$

Make the following combinations of these functions:

$$\begin{cases} v_1(x)=\frac{w_1^+(x)+w_2^+(x)}{2}\equiv\frac{1}{2}\left[\frac{w(x)+w(S_1x)}{2}+\frac{w(S_2x)+w(S_3x)}{2}\right],\\ v_2(x)=\frac{w_1^+(x)-w_2^+(x)}{2}\equiv\frac{1}{2}\left[\frac{w(x)+w(S_1x)}{2}-\frac{w(S_2x)+w(S_3x)}{2}\right],\\ v_3(x)=\frac{w_1^-(x)+w_2^-(x)}{2}\equiv\frac{1}{2}\left[\frac{w(x)-w(S_1x)}{2}+\frac{w(S_2x)-w(S_3x)}{2}\right],\\ v_4(x)=\frac{w_1^-(x)-w_2^-(x)}{2}\equiv\frac{1}{2}\left[\frac{w(x)-w(S_1x)}{2}-\frac{w(S_2x)-w(S_3x)}{2}\right]. \end{cases} \tag{9}$$

Note that the conditions $w(x)|_{\partial\Pi}=0\Rightarrow w(I_jx)|_{\partial\Pi}=0, j=1,2,3$ implies $v(I_jx)|_{\partial\Pi}=0, j=1,2,3,4$, where $\partial\Pi$ is a boundary of the domain Π. Consider the following numbers:

$$\varepsilon_1=a_0+a_1+a_2+a_3,\quad \varepsilon_2=a_0+a_1-a_2-a_3,$$
$$\varepsilon_3=a_0-a_1+a_2-a_3,\quad \varepsilon_4=a_0-a_1-a_2+a_3.$$

It is easy to show that if $w_{k,m}(x)=X_k(x_1)Y_m(x_2)$ are eigenfunctions of the problem (7), (8), then from (9) we obtain the following system

1) $v_{k,m,1}(x)=X_{2k-1}(x_1)\cdot Y_{2m-1}(x_2), k,m=1,2,...;$

2) $v_{k,m,2}(x)=X_{2k-1}(x_1)\cdot Y_{2m}(x_2), k,m=1,2,...;$

3) $v_{k,m,3}(x) = X_{2k}(x_1) \cdot Y_{2m-1}(x_2)$, $k, m = 1, 2, ...$;

4) $v_{k,m,4}(x) = X_{2k}(x_1) \cdot Y_{2m}(x_2)$, $k, m = 1, 2, ...$

The following statement is true.

Theorem 1. *Let $a_j \in R$ be such that $\varepsilon_j \neq 0, j = \overline{1,4}$ and let $w_{k,m}(x) = X_k(x_1) \cdot Y_m(x_2)$ be eigenfunctions of the problem (7), (8) and $\mu_{k,m}$ be corresponding eigenvalues. Then system of the functions $v_{k,m,j}(x_1, x_2), k, m = 1, 2, ..., j = \overline{1,4}$ is eigenfunctions, and*

$$\lambda_{k,m,1} = \varepsilon_1 \mu_{2k-1,\,2m-1}, \quad \lambda_{k,m,2} = \varepsilon_2 \mu_{2k-1,2m},$$
$$\lambda_{k,m,3} = \varepsilon_3 \mu_{2k,2m-1}, \quad \lambda_{k,m,4} = \varepsilon_4 \mu_{2k,2m}, \quad k, m = 1, 2, ...$$

are corresponding eigenvalues of the problem (5), (6).

Proof. The proof of the theorem is carried out by direct application of the operator L to the functions $v_{k,m,j}(x_1, x_2), j = \overline{1,4}$. Recall that

$$v_{k,m,j}(x_1, x_2) = X_k(x_1) \cdot Y_m(x_2),$$
$$X_k(x_1) = \sqrt{\frac{2}{p}} \sin \frac{k\pi}{p} x_1, \quad Y_m(x_2) = \sqrt{\frac{2}{q}} \sin \frac{m\pi}{q} x_2,$$
$$\nu_k = \left(\frac{k\pi}{p}\right)^2, \quad \sigma_m = \left(\frac{m\pi}{q}\right)^2, \quad \mu_{k,m} = \nu_k + \sigma_m.$$

Moreover, we note that

$$\sin \frac{k\pi}{p}(p - x_1) = (-1)^{k+1} \sin \frac{k\pi}{p} x_1, \quad \sin \frac{m\pi}{q}(q - x_2) = (-1)^{m+1} \sin \frac{m\pi}{q} x_2.$$

Let $j = 1$. Then

$$v_{k,m,1}(p - x_1, x_2) = \sin \frac{(2k-1)\pi}{p}(p - x_1) \sin \frac{(2m-1)\pi}{q} x_2 = X_{2k-1}(x_1) Y_{2m-1}(x_2),$$
$$v_{k,m,1}(x_1, q - x_2) = \sin \frac{(2k-1)\pi}{p} x_1 \sin \frac{(2m-1)\pi}{q}(q - x_2) = X_{2k-1}(x_1) Y_{2m-1}(x_2),$$
$$v_{k,m,1}(p - x_1, q - x_2) = \sin \frac{(2k-1)\pi}{p}(p - x_1) \sin \frac{(2m-1)\pi}{q}(q - x_2) = X_{2k-1}(x_1) Y_{2m-1}(x_2).$$

From these equalities, as well as from the equality $(\sin \lambda t)'' = -\lambda^2 \sin \lambda t$, applying the operator $-L$ to the function $u_{k,m,1}(x)$, we have

$$\begin{aligned} L v_{k,m,1}(x) &= a_0 X''_{2k-1}(x_1) Y_{2m-1}(x_2) + a_0 X_{2k-1}(x_1) Y''_{2m-1}(x_2) \\ &+ a_1 X''_{2k-1}(p - x_1) Y_{2m-1}(x_2) + a_1 X_{2k-1}(p - x_1) Y''_{2m-1}(x_2) \\ &+ a_2 X''_{2k-1}(x_1) Y_{2m-1}(q - x_2) + a_2 X_{2k-1}(x_1) Y''_{2m-1}(q - x_2) \\ &+ a_3 X''_{2k-1}(p - x_1) Y_{2m-1}(q - x_2) + a_3 X_{2k-1}(p - x_1) Y''_{2m-1}(q - x_2) \\ &= (-a_0\nu_k - a_0\sigma_m - a_1\nu_k + a_1\sigma_m - a_2\nu_k + a_2\sigma_m + a_3\nu_k + a_3\sigma_m)\, X_{2k-1}(x_1) Y_{2m-1}(x_2) \\ &= [-a_0(\nu_k + \sigma_m) - a_1(\nu_k + \sigma_m) - a_2(\nu_k + \sigma_m) - a_3(\nu_k + \sigma_m)]\, X_{2k-1}(x_1) Y_{2m-1}(x_2) \\ &= -(\nu_k + \sigma_m)(a_0 + a_1 + a_2 + a_3)\, X_{2k-1}(x_1) Y_{2m-1}(x_2) \\ &= -\mu_{k,m} \varepsilon_1 X_{2k-1}(x_1) Y_{2m-1}(x_2) = -\lambda_{k,m,1} v_{k,m,1}(x). \end{aligned}$$

Thus, for the function $v_{k,m,1}(x)$ equality $L v_{k,m,1}(x) = -\lambda_{k,m,1} v_{k,m,1}(x)$ holds, i.e. $v_{k,m,1}(x)$ is the eigenfunction of the operator $-L$, and $\lambda_{k,m,1}$ are the corresponding eigenvalues. For the other functions $v_{k,m,j}(x_1, x_2), j = \overline{2,4}$, the proof is carried out in a similar way. Theorem is proved. □

Since $\bigcup_{j=1}^{4} v_{k,m,j}(x_1, x_2) = \{w_{k,m}(x_1, x_2)\}_{k,m=1}^{\infty}$, Theorem 1 yields

Corollary 1. *System $v_{k,m,j}(x_1, x_2), j = \overline{1,4}$ forms an orthonormal basis in the space $L_2(\Pi)$.*

3. UNIQUENESS OF SOLUTION TO THE PROBLEM ID

The following theorem holds.

Theorem 2. *Let $\varepsilon_j > 0, j = \overline{1,4}$. If a solution to the ID problem exists, then it is unique.*

Proof. Let us show that the homogeneous problem ($\varphi(x) = \psi(x) = 0$) has only a trivial solution. Suppose the opposite. Let there be two solutions $\{u_1(t,x), f_1(x)\}$ and $\{u_2(t,x), f_2(x)\}$ to the problem ID. Denote $\tilde{u}(t,x) = u_1(t,x) - u_2(t,x)$ and $\tilde{f}(x) = f_1(x) - f_2(x)$. Then functions $\tilde{u}(t,x)$ and $\tilde{f}(x)$ satisfy the equation

$$\frac{\partial \tilde{u}(t,x)}{\partial t} = L_x \tilde{u}(t,x) + \tilde{f}(x) \tag{10}$$

and conditions

$$\tilde{u}(0,x) = 0, \quad \tilde{u}(T,x) = 0, \quad x \in \Pi, \tag{11}$$

$$\tilde{u}(t,x) = 0, \quad (t,x) \in \partial\Pi \times [0,T]. \tag{12}$$

Now let the functions $\tilde{u}(t,x)$, $\tilde{f}(x)$ be solutions of the problem (10)–(12). Consider the functions

$$\tilde{u}_{k,m}^{(j)}(t) = \left(\tilde{u}(t,x),\, v_{k,m}^{(j)}(x)\right)_{L_2(\Pi)}, \quad j = \overline{1,4}, \tag{13}$$

$$\tilde{f}_{k,m}^{(j)} = \left(\tilde{f}(x),\, v_{k,m}^{(j)}(x)\right)_{L_2(\Pi)}, \quad j = \overline{1,4}, \tag{14}$$

where $(\xi,\eta)_{L_2(\Pi)}$ means scalar product in the space $L_2(\Pi)$, i.e.

$$(\xi,\eta)_{L_2(\Pi)} = \iint\limits_{\Pi} \xi(x_1,x_2)\eta(x_1,x_2)dx_1dx_2.$$

Differentiating the left and right sides of the equality (13) with respect to t, and taking into account (10), and the conditions (11) and (12), to find unknown functions $\tilde{u}_{k,m}^{(j)}(t), j = \overline{1,4}$ and constants $\tilde{f}_{k,m}^{(j)}, j = \overline{1,4}$, we get the following problems:

$$\frac{du_{k,m}^{(j)}(t)}{dt} + \lambda_{k,m}^{(j)} u_{k,m}^{(j)}(t) = f_{k,m}^{(j)}, \quad j = \overline{1,4}, \quad t \in (0,T), \tag{15}$$

$$\tilde{u}_{k,m}^{(j)}(0) = 0, \quad \tilde{u}_{k,m}^{(j)}(T) = 0, \quad i,j = \overline{1,4}. \tag{16}$$

General solution of the equation (15) has the form

$$\tilde{u}_{k,m}^{(j)}(t) = C_{k,m}^{(j)} e^{-\lambda_{k,m}^{(j)} t} + \frac{f_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}}, \quad j = \overline{1,4}.$$

Further, satisfying the condition (16) by this solution, concerning $C_{k,m}^{(j)}$ and $\tilde{f}_{k,m}^{(j)}$, we obtain the system of the form

$$C_{k,m}^{(j)} + \frac{\tilde{f}_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}} = 0, \quad C_{k,m}^{(j)} e^{-\lambda_{k,m}^{(j)} T^{\alpha}} + \frac{\tilde{f}_{k,m}^{(j)}}{\lambda_{k,m}^{(j)}} = 0,$$

which has only a trivial solution, i.E. $C_{k,m}^{(j)} = 0$, $\tilde{f}_{k,m}^{(j)} = 0$. Then $\tilde{u}_{k,m}^{(j)}(t) = 0$, and from (13) and (14) it follows that the functions $\tilde{u}(t,x), \tilde{f}(x)$ are orthogonal to all elements of the system $v_{k,m}^{(j)}(x)$, which is complete in $L_2(\Pi)$. Thus, $\tilde{u}(t,x) = 0$, $\tilde{f}(x) = 0$, i.e.

$$\tilde{u}(t,x) = 0 \Leftrightarrow u_1(t,x) = u_2(t,x), \quad \tilde{f}(x) = 0 \Leftrightarrow f_1(x) = f_2(x).$$

Uniqueness of solution to the problem ID is proved. □

4. EXISTENCE OF SOLUTION TO THE PROBLEM ID

The following statement is true.

Theorem 3. *Let* $\varepsilon_j > 0,\ j = \overline{1,4},\ \varphi(x),\ \psi(x) \in C^3(\bar{\Pi}),\ \frac{\partial^4 \varphi(x)}{\partial x_r^3 \partial x_s},\ \frac{\partial^4 \psi(x)}{\partial x_r x_s^3} \in C(\bar{\Pi}),\ r, s = 1, 2$ *and functions* $\varphi(x),\ \psi(x),\ \frac{\partial^2 \varphi(x)}{\partial x_r \partial x_s}, \frac{\partial^2 \psi(x)}{\partial x_r \partial x_s},\ r, s = 1, 2$ *satisfy the boundary value condition (4). Then solution to the problem ID exists, and is represented in the form*

$$u(t,x) = \varphi(x) - \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} \frac{1 - e^{-\lambda_{k,m,j} t}}{1 - e^{-\lambda_{k,m,j} T}} \left[\psi_{k,m,j} - \varphi_{k,m,j}\right] v_{k,m,j}(x), \tag{17}$$

$$f(x) = -L\varphi(x) - \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} \lambda_{k,m,j} \left(\frac{\varphi_{k,m,j} - \psi_{k,m,j}}{1 - e^{-\lambda_{k,m,j} T}} \right) v_{k,m,j}(x). \tag{18}$$

Proof. Since system of eigenfunctions $v_{k,m,j}(x_1, x_2), j = \overline{1,4}$ of the problem (5), (6) forms orthonormal basis in $L_2(\Pi)$, then solution $(u(x,t), f(x))$ of the problem (1)–(4) can be represented as a series expansion in this system, that is

$$u(t,x) = \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} u_{k,m,j}(t) v_{k,m,j}(x), \tag{19}$$

$$f(x) = \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} f_{k,m,j} v_{k,m,j}(x), \tag{20}$$

where $u_{k,m,j}(t)$ are unknown functions, $f_{k,m,j}$ are unknown numbers, $k, m \in N,\ j = \overline{1,4}$. Putting (19) and (20) into the equation (1), to find functions $u_{k,m,j}(t)$ and constants $f_{k,m,j}$, we get the following equations

$$\frac{d}{dt} u_{k,m,j}(t) + \lambda_{k,m,j} u_{k,m,j}(t) = f_{k,m,j}, \quad j = \overline{1,4}, t \in (0,T),$$

where $\lambda_{k,m,j} = \varepsilon_j \mu_{k,m}$. Solving these equations, we have

$$u_{k,m,j}(t) = C_{k,m,j} e^{-\lambda_{k,m,j} t} + \frac{f_{k,m,j}}{\lambda_{k,m,j}}, \quad j = \overline{1,4}, \tag{21}$$

where $C_{k,m,j}$ are unknown numbers. To find these constants, we use conditions (2). For this, we assume that the functions $\varphi(x)$ and $\psi(x)$ are expanded in a Fourier series in the system $v_{k,m,j}(x_1, x_2), j = \overline{1,4}$, i.e.

$$\varphi(x) = \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} \varphi_{k,m,j} v_{k,m,j}(x), \quad \psi(x) = \sum_{k,m=1}^{\infty} \sum_{j=1}^{4} \psi_{k,m,j} v_{k,m,j}(x),$$

where

$$\varphi_{k,m,j} = \left(\varphi(x), v_{k,m,j}(x)\right)_{L_2(\Pi)}, \quad \psi_{k,m,j} = \left(\psi(x), v_{k,m,j}(x)\right)_{L_2(\Pi)}, \quad j = \overline{1,4}.$$

Then conditions (2) concerning $u_{k,m,j}(t)$ have the form

$$u_{k,m,j}(0) = \varphi_{k,m,j}, \quad u_{k,m,j}(T) = \psi_{k,m,j},\ j = \overline{1,4}.$$

Taking into account it, from (21) we obtain that

$$C_{k,m,j} + \frac{f_{k,m,j}}{\lambda_{k,m,j}} = \varphi_{k,m,j}, \quad C_{k,m,j} e^{-\lambda_{k,m,j} T} + \frac{f_{k,m,j}}{\lambda_{k,m,j}} = \psi_{k,m,j}, \quad j = \overline{1,4}.$$

Hence, we find $C_{k,m,j}$ and $f_{k,m,j}, j = \overline{1,4}$,

$$C_{k,m,j} = \frac{\varphi_{k,m,j} - \psi_{k,m,j}}{1 - e^{-\lambda_{k,m,j} T}}, \quad j = \overline{1,4}, \tag{22}$$

$$f_{k,m,j}=\lambda_{k,m,j}\left(\varphi_{k,m,j}-C_{k,m,j}\right),\quad j=\overline{1,4}. \tag{23}$$

Putting expressions for $u_{k,m,j}(t)$ and $f_{k,m,j}$ into (19) and (20), we obtain

$$u(t,x)=\varphi(x)+\sum_{k,m=1}^{\infty}\sum_{j=1}^{4}(e^{-\lambda_{k,m,j}t}-1)C_{k,m,j}v_{k,m,j}(x), \tag{24}$$

$$f(x)=\sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}\left(\varphi_{k,m,j}-C_{k,m,j}\right)v_{k,m,j}(x). \tag{25}$$

Thus, we have found a formal form of the solution of the problem in the form of series (24) and (25), where the coefficients $C_{k,m,j}$ are calculated by formulas (22). By direct calculation it is easy to show that the functions $u(t,x)$, $f(x)$, defined by series (24) and (25), satisfy equation (1) and conditions (2)–(4). It remains to prove legality of these actions. For this, we will show that $u(x,t)\in C^{1,2}_{t,x}\left(\bar{Q}\right)$, $f(x)\in C(\bar{\Pi})$. Furthermore, C will mean an arbitrary constant, value of which is not of interest to us. Let $g(x_1,x_2)$ be a function that satisfies conditions of the theorem. Then

$$\int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2=\int_0^q\int_0^p g(x_1,x_2)\frac{d}{dx_1}\left(-\frac{p}{k\pi}cos\frac{k\pi}{p}x_1\right)\sin\frac{m\pi}{q}x_2dx_1dx_2$$

$$=-\frac{p}{k\pi}g(x_1,x_2)\cos\frac{k\pi}{p}x_1\Big|_{x_1=0}^{x_1=p}+\frac{p}{k\pi}\int_0^q\int_0^p\frac{\partial g(x_1,x_2)}{\partial x_1}\cos\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2$$

$$=\left(\frac{p}{k\pi}\right)^2\int_0^q\int_0^p\frac{\partial g(x_1,x_2)}{\partial x_1}\frac{d}{dx_2}\left(\sin\frac{k\pi}{p}x_1\right)\sin\frac{m\pi}{q}x_2dx_1dx_2$$

$$=-\left(\frac{p}{k\pi}\right)^2\int_0^q\int_0^p\frac{\partial^2 g(x_1,x_2)}{\partial x_1^2}\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2$$

$$=-\left(\frac{p}{k\pi}\right)^3\int_0^q\int_0^p\frac{\partial^3 g(x_1,x_2)}{\partial x_1^3}\cos\frac{k\pi}{p}x_1\frac{d}{dx_1}\left(-\frac{q}{m\pi}\cos\frac{m\pi}{q}x_2\right)dx_1dx_2$$

$$=\left(\frac{p}{k\pi}\right)^3\frac{q}{m\pi}\int_0^q\int_0^p\frac{\partial^4 g(x_1,x_2)}{\partial x_1^3\partial x_2}\cos\frac{k\pi}{p}x_1\cos\frac{m\pi}{q}x_2dx_1dx_2=\frac{C}{k^3m}g^{3,1}_{k,m},$$

where

$$g^{3,1}_{k,m}=\int_0^q\int_0^p\frac{\partial^4 g(x_1,x_2)}{\partial x_1^3\partial x_2}\cos\frac{k\pi}{p}x_1\cos\frac{m\pi}{q}x_2dx_1dx_2. \tag{26}$$

Therefore, we have the equality

$$\int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2=\frac{C}{k^3m}g^{3,1}_{k,m}.$$

In a similar way, we obtain the equality:

$$\int_0^q\int_0^p g(x_1,x_2)\sin\frac{k\pi}{p}x_1\sin\frac{m\pi}{q}x_2dx_1dx_2=\frac{C}{km^3}g^{1,3}_{k,m},$$

where

$$g_{k,m}^{1,3} = \int_0^q \int_0^p \frac{\partial^4 g(x_1,x_2)}{\partial x_1 \partial x_2^3} \cos\frac{k\pi}{p}x_1 \cos\frac{m\pi}{q}x_2 dx_1 dx_2. \quad (27)$$

Now we will use the obtained equalities to estimate $C_{k,m,j}$. Suppose that $j=1$. Integrating by parts the expressions for the coefficients $C_{k,m,1}$ from formulas (22) and (23), we obtain

$$C_{k,m,1} = \frac{\varphi_{k,m,1} - \psi_{k,m,1}}{1 - e^{-\lambda_{k,m,1}T}} = \frac{C}{1 - e^{-\lambda_{k,m,1}T}}$$
$$\times \int_0^q \int_0^p [\varphi(x_1,x_2) - \psi(x_1,x_2)] \sin\frac{(2k-1)\pi}{p}x_1 \sin\frac{(2m-1)\pi}{q}x_2 dx_1 dx_2$$
$$= \frac{C}{\left(1 - e^{-\lambda_{k,m,1}T}\right)(2k-1)^3(2m-1)} \left(\varphi_{k,m,1}^{3,1} - \psi_{k,m,1}^{3,1}\right) \quad (28)$$

and

$$C_{k,m,1} = \frac{C}{\left(1 - e^{-\lambda_{k,m,1}T}\right)(2k-1)(2m-1)^3} \left(\varphi_{k,m,1}^{1,3} - \psi_{k,m,1}^{1,3}\right), \quad (29)$$

where the coefficients $\varphi_{k,m,1}^{3,1},\ \psi_{k,m,1}^{3,1},\ \varphi_{k,m,1}^{1,3},\ \psi_{k,m,1}^{1,3}$ are defined as in formulas (26) and (27). Similarly, we can write the coefficients $C_{k,m,j}, j = 2,3,4$. Further, for the function $v_{k,m,1}(x)$ we have

$$\frac{\partial^2 v_{k,m,1}(x)}{\partial x_1^2} = -\left(\frac{(2k-1)\pi}{p}\right)^2 \sin\frac{(2k-1)\pi}{p}x_1 \sin\frac{(2m-1)\pi}{q}x_2,$$
$$\frac{\partial^2 v_{k,m,1}(x)}{\partial x_2^2} = -\left(\frac{(2m-1)\pi}{q}\right)^2 \sin\frac{(2k-1)\pi}{p}x_1 \sin\frac{(2m-1)\pi}{q}x_2.$$

Estimate the coefficients $(2k-1)^2 C_{k,m,1},\ (2m-1)^2 C_{k,m,1}$. By using representation (28), we have

$$(2k-1)^2 |C_{k,m,1}| = \left| \frac{C(2k-1)^2}{\left(1 - e^{-\lambda_{k,m,1}T}\right)(2k-1)^3(2m-1)} \left(\varphi_{k,m,1}^{3,1} - \psi_{k,m,1}^{3,1}\right) \right|$$
$$\leq \frac{C}{(2k-1)(2m-1)} \left(\left|\varphi_{k,m,1}^{3,1}\right| + \left|\psi_{k,m,1}^{3,1}\right| \right).$$

Similarly, from (29) we obtain

$$(2m-1)^2 |C_{k,m,1}| = \left| \frac{C(2m-1)^2}{\left(1 - e^{-\lambda_{k,m,1}T}\right)(2k-1)(2m-1)^3} \left(\varphi_{k,m,1}^{1,3} - \psi_{k,m,1}^{1,3}\right) \right|$$
$$\leq \frac{C}{(2k-1)(2m-1)} \left(\left|\varphi_{k,m,1}^{1,3}\right| + \left|\psi_{k,m,1}^{1,3}\right| \right).$$

Similar estimates are valid for the coefficients $(2k-1)^2 C_{k,m,j}, (2m-1)^2 C_{k,m,j}, j = 2,3,4$. Since the system $w_{k,m}(x_1,x_2) = \cos\frac{(2k-1)\pi}{p}x_1 \cos\frac{(2m-1)\pi}{q}x_2$ is orthonormal and $\frac{\partial^4\varphi(x_1,x_2)}{\partial x_1^3 \partial x_2}, \frac{\partial^4\psi(x_1,x_2)}{\partial x_1^3 \partial x_2} \in L_2(\Pi)$, then applying the Cauchy–Schwarz and Bessel inequalities, we have

$$\sum_{k,m=1}^{\infty} (2k-1)^2 |C_{k,m,1}| \leq \sum_{k,m=1}^{\infty} \frac{\left(\left|\varphi_{k,m,1}^{3,1}\right| + \left|\psi_{k,m,1}^{3,1}\right| \right)}{(2k-1)(2m-1)}$$
$$\leq \sqrt{\sum_{k,m=1}^{\infty} \frac{1}{(2k-1)^2(2m-1)^2}} \left(\sqrt{\sum_{k,m=1}^{\infty} \left|\varphi_{k,m,1}^{3,1}\right|^2} + \sqrt{\sum_{k,m=1}^{\infty} \left|\psi_{k,m,1}^{3,1}\right|^2} \right) < \infty,$$

$$\sum_{k,m=1}^{\infty}(2m-1)^2\left|C_{k,m,1}\right| \leq \sum_{k,m=1}^{\infty}\frac{\left(\left|\varphi_{k,m,1}^{1,3}\right|+\left|\psi_{k,m,1}^{1,3}\right|\right)}{(2k-1)(2m-1)}$$

$$\leq \sqrt{\sum_{k,m=1}^{\infty}\frac{1}{(2k-1)^2(2m-1)^2}}\left(\sqrt{\sum_{k,m=1}^{\infty}\left|\varphi_{k,m,1}^{1,3}\right|^2}+\sqrt{\sum_{k,m=1}^{\infty}\left|\psi_{k,m,1}^{1,3}\right|^2}\right)<\infty.$$

Convergence of the series corresponding to the coefficients $(2k-1)^2C_{k,m,j}$, $(2m-1)^2C_{k,m,j}, j=2,3,4$, is proved similarly. Then the series, obtained from (22) by differentiation twice with respect to x_1 converges absolutely and uniformly in the domain $\bar{Q}$. Consequently, the sum of this series represents a continuous function in the domain $\bar{Q}$, i.e $u_{x_1x_1}(t,x)\in C\left(\bar{Q}\right)$. In a similar way it can be shown that $u_{x_2x_2}(t,x)\in C\left(\bar{Q}\right)$ and $u_t(t,x)\in C\left(\bar{Q}\right)$. Now let us prove that $f(x)\in C\left(\bar{Q}\right)$. For this, due to (25), it is necessary to study convergence of the series

$$\sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}\varphi_{k,m,j}v_{k,m,j}(x),\ \sum_{k,m=1}^{\infty}\sum_{j=1}^{4}\lambda_{k,m,j}C_{k,m,j}v_{k,m,j}(x).$$

We have already shown convergence of the second series, and convergence of the first series is proved in a similar way. Indeed, for example, in the case $j=1$ we have

$$\varphi_{k,m,1}=\frac{2}{\sqrt{pq}}\int_0^q\int_0^p\varphi(x_1,x_2)\sin\frac{(2k-1)\pi}{p}x_1\sin\frac{(2m-1)\pi}{q}x_2dx_1dx_2$$

$$=\frac{2}{\sqrt{pq}}\left(\frac{p}{(2k-1)\pi}\right)^3\frac{q}{(2m-1)\pi}\int_0^q\int_0^p\frac{\partial^4\varphi(x_1,x_2)}{\partial x_1^3\partial x_2}\cos\frac{(2k-1)\pi}{p}x_1$$

$$\times\cos\frac{(2m-1)\pi}{q}x_2dx_1dx_2=\frac{C}{(2k-1)^3(2m-1)}\varphi_{k,m}^{3,1}.$$

Consequently, we have that

$$\varphi_{k,m,1}=\frac{C}{(2k-1)^3(2m-1)}\varphi_{k,m,1}^{3,1},\quad \varphi_{k,m,1}=\frac{C}{(2k-1)(2m-1)^3}\varphi_{k,m,1}^{1,3}.$$

From this, we obtain absolute and uniform convergence of the series

$$\sum_{k,m=1}^{\infty}\lambda_{k,m,1}\varphi_{k,m,1}v_{k,m,1}(x).$$

In a similar way, we can show convergence of the series

$$\sum_{k,m=1}^{\infty}\lambda_{k,m,j}\varphi_{k,m,j}v_{k,m,j}(x),\quad j=2,3,4.$$

Then the series (25) converges absolutely and uniformly in $\bar{\Pi}$, i.e $f(x)\in C\left(\bar{\Pi}\right)$. Further, by using equalities (22) and (23), we obtain representations of solutions to the problem in the form (17) and (18). Theorem is proved. □

5. CONCLUSIONS

We studied the inverse boundary value problem ID for a nonlocal parabolic equation with following assumption

$$\varepsilon_j>0,\quad j=\overline{1,4},\quad \varphi(x),\psi(x)\in C^3(\bar{\Pi}),\quad \frac{\partial^4\varphi(x)}{\partial x_r^3\partial x_s},\quad \frac{\partial^4\psi(x)}{\partial x_r x_s^3}\in C(\bar{\Pi}),\quad r,s=1,2.$$

If these conditions are fulfulled, and functions $\varphi(x), \psi(x), \frac{\partial\varphi(x)}{\partial x_r \partial x_s}, \frac{\partial\psi(x)}{\partial x_r \partial x_s}, r, s = 1, 2$ satisfied the boundary value conditions (3), then the inverse boundary value problem ID is uniquely solvable and this solution is represented in the form of the Fourier series (20) and (21) in the domain Q. We also proposed a method of reducing the spectral problem for the nonlocal Laplace equation to the same spectral problem for the classical Laplace equation, which can be applied to solve similar problems.

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